Deformations of Azumaya Algebras with Quadratic Pair

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Quadratic Pairs

Working over a scheme X.

Definition (KMRT, Calmés+Fasel)

For an Azumaya $\mathcal{O}_X$–algebra $\mathcal{A}$, a quadratic pair on $\mathcal{A}$ is (σ, f) where

- $\sigma \colon \mathcal{A} \to \mathcal{A}$ *is an orthogonal involution, and*
- $f \colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}_X$ *is an $\mathcal{O}_X$–linear map satisfying*

$$f(a + \sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(a)$$

for all sections $a \in \mathcal{A}$.

Related to algebraic groups of type D in arbitrary characteristic/over arbitrary schemes. Ex., every adjoint group of type D_n is some $\mathbf{PGO}^{+}_{(\mathcal{A},\sigma,f)}$ with $\deg(\mathcal{A}) = 2n$.

Deformation Problems

- $X \hookrightarrow X'$ closed embedding defined by $\mathcal{J} \subset \mathcal{O}_{X'}$, $\mathcal{J}^2 = 0$.
- G' an algebraic group on X'.
- $\mathcal{P}$ a $G'|_X$–torsor (on X).

When does $\mathcal{P}$ come from X'?

Theorem (Illusie, 1972)

(i) *$\exists \mathcal{P}'$ a G'–torsor on X' such that $\mathcal{P}'|_X \cong \mathcal{P}$ if and only if* $\mathrm{obs}(\mathcal{P}) \in H^2(X, \mathfrak{Lie}(\mathcal{A}ut(\mathcal{P})) \otimes_{\mathcal{O}_X} \mathcal{J})$ *is zero.*

(ii) *If* $\mathrm{obs}(\mathcal{P}) = 0$, *all such $\mathcal{P}'$ are classified by* $H^1(X, \mathfrak{Lie}(\mathcal{A}ut(\mathcal{P})) \otimes_{\mathcal{O}_X} \mathcal{J})$

(iii) *For a fixed $\mathcal{P}'$,*

$$\mathrm{Ker}\,(\mathcal{A}ut(\mathcal{P}') \xrightarrow{\text{res}} \mathcal{A}ut(\mathcal{P})) \cong H^0(X, \mathfrak{Lie}(\mathcal{A}ut(\mathcal{P})) \otimes_{\mathcal{O}_X} \mathcal{J}).$$

Relative Obstructions

If $\varphi\colon G' \to H'$ is a morphism of groups on X', we get

$$\begin{aligned} H^1(X, G') &\to H^1(X, H') \\ [\mathcal{P}] &\mapsto [\varphi_*(\mathcal{P})] \end{aligned}$$

as well as

$$\begin{aligned} H^2(X, \mathfrak{Lie}(\mathcal{A}ut(\mathcal{P})) \otimes_{\mathcal{O}_X} \mathcal{J}) &\to H^2(X, \mathfrak{Lie}(\mathcal{A}ut(\varphi_*(\mathcal{P}))) \otimes_{\mathcal{O}_X} \mathcal{J}) \\ \mathrm{obs}(\mathcal{P}) &\mapsto \mathrm{obs}(\varphi_*(\mathcal{P})). \end{aligned}$$

Question

We consider $\mathbf{PGO}_{2n} \hookrightarrow \mathbf{PGL}_{2n}$.

$$\begin{aligned} H^1(X, \mathbf{PGO}_{2n}) &\to H^1(X, \mathbf{PGL}_{2n}) \\ [(\mathcal{A}, \sigma, f)] &\mapsto [\mathcal{A}]. \end{aligned}$$

Is there an example where $(\mathcal{A}, \sigma, f)$ does not deform, but $\mathcal{A}$ does?

Convenient Setup

Let $\mathbb{K}$ be a field, $(C, \mathfrak{m})$, $(C', \mathfrak{m}')$ Artinian local $\mathbb{K}$–algebras with residue field $\mathbb{K}$.

A surjection $C' \twoheadrightarrow C$ with kernel $J \subset C'$ such that $J \cdot \mathfrak{m}' = 0$ is a *small extension*.

$$\begin{tikzcd}
X_C \arrow[rr, hook] \arrow[dr, shift left] \arrow[dd] & & X_{C'} \arrow[dl, shift left] \arrow[dd] \\
& X \arrow[ul, shift left] \arrow[ur, shift left] \arrow[dd] & \\
\mathrm{Spec}(C) \arrow[rr, hook] \arrow[dr, shift left] & & \mathrm{Spec}(C') \arrow[dl, shift left] \\
& \mathrm{Spec}(\mathbb{K}) \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

Tangent-Obstruction Theory

Let $\mathcal{P}$ be a G–torsor on X.

$$\begin{aligned} D\colon \mathrm{Art}_{\mathbb{K}} &\to \mathfrak{Sets} \\ C &\mapsto \{\mathcal{P}' \mid \mathcal{P}' \text{ is a } G|_{X_C}\text{–torsor}, \mathcal{P}'|_X \cong \mathcal{P}\}/\sim \end{aligned}$$

For any small extension $C' \twoheadrightarrow C$ with kernel J, we get

$$H^1(X, \mathfrak{Lie}(\mathcal{A}ut(\mathcal{P}))) \otimes_{\mathbb{K}} J \to D(C') \xrightarrow{\text{res}} D(C) \to H^2(X, \mathfrak{Lie}(\mathcal{A}ut(\mathcal{P}))) \otimes_{\mathbb{K}} J,$$

a short exact sequence of pointed sets.

- The cohomology sets are always over X.

Tangent-Obstruction Theory

Given $\varphi\colon G \to H$, and $\mathcal{P}$ a G–torsor. Letting

$$\begin{aligned} D\colon \mathrm{Art}_{\mathbb{K}} &\to \mathfrak{Sets} \\ C &\mapsto \{\text{Deformations of } \mathcal{P} \text{ to } X_C\}/\sim \end{aligned}$$

$$\begin{aligned} F\colon \mathrm{Art}_{\mathbb{K}} &\to \mathfrak{Sets} \\ C &\mapsto \{\text{Deformations of } \varphi_*(\mathcal{P}) \text{ to } X_C\}/\sim \end{aligned}$$

we have

$$\begin{tikzcd} H^1(X, \text{``}\mathcal{P}\text{''}) \otimes_{\mathbb{K}} J \arrow[r] \arrow[d] & D(C') \arrow[r] \arrow[d] & D(C) \arrow[r] \arrow[d] & H^2(X, \text{``}\mathcal{P}\text{''}) \otimes_{\mathbb{K}} J \arrow[d] \\ H^1(X, \text{``}\varphi_*(\mathcal{P})\text{''}) \otimes_{\mathbb{K}} J \arrow[r] & F(C') \arrow[r] & F(C) \arrow[r] & H^2(X, \text{``}\varphi_*(\mathcal{P})\text{''}) \otimes_{\mathbb{K}} J \end{tikzcd}$$

Some Relatively Unobstructed Cases

If $2 \in \mathcal{O}_X^\times$ or if $\deg(A) = 2$, then

$$(\mathcal{A}, \sigma, f) \text{ deforms} \Leftrightarrow \mathcal{A} \text{ deforms}.$$

In general,

$$\begin{aligned} \mathfrak{Lie}(\mathbf{PGL}_{\mathcal{A}}) &\cong \mathcal{A}/\mathcal{O}_X \\ \mathfrak{Lie}(\mathbf{PGO}_{(\mathcal{A},\sigma,f)}) &\cong \{x \in \mathcal{A}/\mathcal{O}_X \mid x + \overline{\sigma}(x) = 0, f \circ \mathrm{ad}(x)|_{\mathcal{S}ym_{\mathcal{A},\sigma}} = 0\}. \end{aligned}$$

If $2 \in \mathcal{O}_X^\times$, $\mathfrak{Lie}(\mathbf{PGO}_{(\mathcal{A},\sigma,f)}) \cong \mathcal{A}lt_{\mathcal{A},\sigma}$ "$= \{a - \sigma(a) \mid a \in \mathcal{A}\}$", and

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{Lie}(\mathbf{PGO}_{(\mathcal{A},\sigma,f)}) \arrow[r, hook] & \mathfrak{Lie}(\mathbf{PGL}_{\mathcal{A}}) \arrow[l, bend left, "\frac{1}{2}(\mathrm{Id} - \sigma)"] \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

is a splitting.

Some Relatively Unobstructed Cases

Over any scheme when $\deg(A) = 2$,

$$\mathfrak{Lie}(\mathbf{PGL}_2) \cong \mathrm{M}_2(\mathcal{O}_X)/\mathcal{O}_X \cong \left\{ \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \right\}$$

$$\mathfrak{Lie}(\mathbf{PGO}_{(\mathrm{M}_2(\mathcal{O}_X),\sigma_2,f_2)}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \right\}.$$

So,

$$\mathfrak{Lie}(\mathbf{PGL}_2) \cong \mathfrak{Lie}(\mathbf{PGO}_{(\mathrm{M}_2(\mathcal{O}_X),\sigma_2,f_2)}) \oplus N.$$

and this splitting is stabilized by $\mathbf{PGO}_2$, so it twists to

$$\mathfrak{Lie}(\mathbf{PGL}_{\mathcal{A}}) \cong \mathfrak{Lie}(\mathbf{PGO}_{(\mathcal{A},\sigma,f)}) \oplus N'.$$

A Relatively Obstructed Example

The next case is $\deg(\mathcal{A}) = 4$ and $2 \notin \mathcal{O}_X^\times$. Here, we construct an example where

$$(\mathcal{A}, \sigma, f) \text{ does not deform, but } \mathcal{A} \text{ deforms.}$$

Ingredients:

- Norm equivalence $A_1^2 \xrightarrow{\sim} D_2$.
- Igusa surface X over $\mathbb{K}$, $\mathrm{char}(\mathbb{K}) = 2$.
- The example is a biquaternion algebra on $X_{k[x]/\langle x^2 \rangle}$ which does not deform to $X_{k[x]/\langle x^3 \rangle}$.

Norm Equivalence

Joint work with Philippe Gille and Erhard Neher.
Two stacks:

$$\begin{aligned} A_1^2 &\to \mathfrak{Sch}_X \\ (T' \to T, \mathcal{Q}) &\mapsto T \end{aligned} \qquad\qquad \begin{aligned} D_2 &\to \mathfrak{Sch}_X \\ (T, (\mathcal{A}, \sigma, f)) &\mapsto T \end{aligned}$$

(i) $T' \to T$ degree 2 étale cover
(ii) $\mathcal{Q}$ is on $T', \deg(\mathcal{Q}) = 2$

(i) (σ, f) quadratic pair
(ii) $\deg(\mathcal{A}) = 4$.

We have an equivalence of stacks

$$N\colon A_1^2 \xrightarrow{\sim} D_2.$$

$$(T \sqcup T \to T, (\mathcal{Q}_1, \mathcal{Q}_2)) \mapsto (T, (\mathcal{Q}_1 \otimes_{\mathcal{O}_T} \mathcal{Q}_2, \sigma_1 \otimes \sigma_2, f_{\otimes})).$$

Norm Equivalence

Let $X \hookrightarrow X'$ be an infinitesimal thickening.

$$\begin{tikzcd}
(T \to X', \mathcal{Q}') \arrow[r, mapsto, "N"] \arrow[d, mapsto, "\mathrm{res}"] & (X', (\mathcal{A}', \sigma', f')) \arrow[d, mapsto, "\mathrm{res}"] \\
(X \sqcup X \to X, (\mathcal{Q}_1, \mathcal{Q}_2)) \arrow[r, mapsto, "N"] & (X, (\mathcal{Q}_1 \otimes_{\mathcal{O}_X} \mathcal{Q}_2, \sigma_1 \otimes \sigma_2, f_\otimes))
\end{tikzcd}$$

- Étale extensions have trivial deformation theory. $\Rightarrow T = X' \sqcup X'$.
- $\Rightarrow \mathcal{Q}' = (\mathcal{Q}'_1, \mathcal{Q}'_2)$.

$$(\mathcal{Q}_1 \otimes_{\mathcal{O}_X} \mathcal{Q}_2, \sigma_1 \otimes \sigma_2, f_\otimes) \text{ deforms to } X'$$

$$\Updownarrow$$

$$\text{Both } \mathcal{Q}_1 \text{ and } \mathcal{Q}_2 \text{ deform to } X'.$$

Igusa Surface

Let E be an ordinary elliptic curve over $\mathbb{K}$, $\mathbb{K} = \overline{\mathbb{K}}$, $\mathrm{char}(\mathbb{K}) = 2$.

$$E[2] \cong \mu_2 \times \mathbb{Z}/2\mathbb{Z} \text{ and } E[2](\mathbb{K}) = \{0, t\}.$$

$\mathbb{Z}/2\mathbb{Z}$ acts on $E \times_{\mathbb{K}} E$ by

$$(a, b) \mapsto (-a, b + t)$$

This action has a smooth, projective quotient

$$\pi \colon E \times_{\mathbb{K}} E \to X$$

X is an Igusa surface.

- It has trivial canonical bundle, $K_X \cong \mathcal{O}_X$.
- $\dim_{\mathbb{K}}(H^1(X, \mathcal{O}_X)) = 2$, $\dim_{\mathbb{K}}(H^2(X, \mathcal{O}_X)) = 1$.

Igusa Surface

- The Picard scheme of X is non-reduced (Igusa, Jensen, Srinivas+Mehta).

$$\mathrm{Pic}^0_{X/\mathbb{K}} \cong \mu_2 \times E/\langle t \rangle.$$

$\mu_2 = \mathrm{Spec}(k[x]/\langle x^2 \rangle)$, and let $\tau_3 = \mathrm{Spec}(k[x]/\langle x^3 \rangle)$. Consider

$$\begin{tikzcd}
X_{\mu_2} \arrow[rr, hook] \arrow[dr, shift left] & & X_{\tau_3} \arrow[dl] \\
& X \arrow[ul, hook, shift left] &
\end{tikzcd}$$

For $\mathcal{L}$ on X, $\mathcal{L} = (x, y) \in \mathrm{Pic}^0_{X/\mathbb{K}}$,

$$T_{\mathcal{L}}(\mathrm{Pic}^0_{X/\mathbb{K}}) \cong T_x(\mu_2) \oplus T_y(E/\langle t \rangle) \cong H^1(X, \mathcal{O}_X).$$

any $\mathcal{L}'$ with non-trivial $T_x(\mu_2)$ component does not deform to X_{τ_3}.

Sketch of the Construction

Using explicit cocycles, define

$$\mathcal{Q} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{V}) \text{ and } \mathcal{P} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{W})$$

two quaternions on X. Calculate: $\mathcal{Q}(X) = \mathcal{P}(X) = (\mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{P})(X) = \mathbb{K}$.

Proposition

Let $\mathrm{char}(\mathbb{K}) = 2$ *and* X *a smooth, projective surface over* $\mathbb{K}$ *with trivial canonical bundle. Let* $(\mathcal{A}, \sigma, f)$ *be an Azumaya algebra with quadratic pair on* X *such that* $\mathcal{A}(X) \xrightarrow{\mathrm{Id} + \sigma} \mathcal{A}\ell t_{\mathcal{A},\sigma}(X)$ *and* $\mathcal{S}ym_{\mathcal{A},\sigma}(X) \xrightarrow{f} \mathcal{O}_X(X)$ *are both zero. Then,*

$$H^2(X, \mathfrak{Lie}(\mathbf{PGO}_{(\mathcal{A},\sigma,f)})) \to H^2(X, \mathfrak{Lie}(\mathbf{PGL}_{\mathcal{A}}))$$

is zero also.

- $(\mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{P}, \sigma_{\mathcal{Q}} \otimes \sigma_{\mathcal{P}}, f_{\otimes})$ satisfies the proposition.
- $\Rightarrow$ any deformation of $\mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{P}$ to X_{μ_2} itself deforms to X_{τ_3}.

Sketch of the Construction

For $\mathcal{Q}$ (likewise for $\mathcal{P}$)

$$0 \to \mathcal{O}_X \to \mathcal{Q} \to \mathcal{Q}/\mathcal{O}_X \to 0$$

induces

$$\begin{array}{ccccccc} H^2(X, \mathcal{O}_X) & \longrightarrow & H^2(X, \mathcal{Q}) & \longrightarrow & H^2(X, \mathcal{Q}/\mathcal{O}_X) & \longrightarrow & 0 = H^3(X, \mathcal{O}_X) \\ & & \| & & \| & & \\ & & H^2(X, \mathfrak{Lie}(\mathbf{GL}_{\mathcal{V}})) & & H^2(X, \mathfrak{Lie}(\mathbf{PGL}_{\mathcal{Q}})) & & \end{array}$$

Serre dual to

$$0 \to (\mathcal{Q}/\mathcal{O}_X)^\vee(X) \to \mathcal{Q}^\vee(X) \xrightarrow{0} \mathcal{O}_X^\vee(X)$$

- $\Rightarrow H^2(X, \mathcal{Q}) \xrightarrow{\sim} H^2(X, \mathcal{Q}/\mathcal{O}_X)$
- $\Rightarrow \mathcal{Q} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{V})$ deforms if and only if $\mathcal{V}$ deforms.

Sketch of the Construction

$\mathrm{Trd}_{\mathcal{Q}} \colon \mathcal{Q} \to \mathcal{O}_X$ and $\mathrm{Trd}_{\mathcal{P}} \colon \mathcal{P} \to \mathcal{O}_X$ induce

$$\begin{aligned} H^1(X, \mathcal{Q}) &\to H^1(X, \mathcal{O}_X) \\ [\mathcal{V}'] &\mapsto [\det(\mathcal{V}')] \end{aligned} \qquad \begin{aligned} H^1(X, \mathcal{P}) &\to H^1(X, \mathcal{O}_X) \\ [\mathcal{W}'] &\mapsto [\det(\mathcal{W}')]. \end{aligned}$$

By analyzing the module structures of $\mathcal{Q}, \mathcal{P}$, see that these maps have linearly independent images in

$$H^1(X, \mathcal{O}_X) = T_x(\mu_2) \oplus T_y(E/\langle t \rangle)$$

- $\Rightarrow$ some $\mathcal{V}'$ or $\mathcal{W}'$ on X_{μ_2} has obstructed determinant bundle.
- $\Rightarrow$ some $\mathcal{V}'$ or $\mathcal{W}'$ is obstructed
- $\Rightarrow$ some $\mathcal{Q}'$ or $\mathcal{P}'$ is obstructed.

The Example

- Take $\mathcal{Q}'$ on X_{μ_2} deforming $\mathcal{Q}$, obstructed on X_{τ_3}.

$$
\begin{tikzcd}
(\mathcal{Q}' \otimes_{\mathcal{O}_{X_{\mu_2}}} (\mathcal{P}|_{X_{\mu_2}}), \sigma_{\mathcal{Q}'} \otimes (\sigma_{\mathcal{P}})|_{X_{\mu_2}}, f'_{\otimes}) \arrow[dr, mapsto] \arrow[dd, dash, dashed] & & \exists \mathcal{A}'' \arrow[ll, mapsto, bend right] \arrow[dl, mapsto, bend right] \arrow[dd, dash, dashed] \\
& (\mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{P}, \sigma_{\mathcal{Q}} \otimes \sigma_{\mathcal{P}}, f_{\otimes}) \arrow[dd, dash, dashed] & \\
X_{\mu_2} \arrow[rr, hook] & & X_{\tau_3} \\
& X \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

- No deformation $(\mathcal{A}'', \sigma'', f'')$ on X_{τ_3} exists.

Thank You