Deformations of Azumaya Algebras with Quadratic Pair

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Quadratic Pairs

Working over a scheme X.

Definition (KMRT, Calmés+Fasel)

For an Azumaya \mathcal{O}_X -algebra \mathcal{A} , a quadratic pair on \mathcal{A} is (σ, f) where

- $\sigma \colon \mathcal{A} \to \mathcal{A}$ is an orthogonal involution, and
- ullet f: Sym $_{\mathcal{A},\sigma} o \mathcal{O}_X$ is an \mathcal{O}_X -linear map satisfying

$$f(a+\sigma(a))=\mathrm{Trd}_{\mathcal{A}}(a)$$

for all sections $a \in A$.

Related to algebraic groups of type D in arbitrary characteristic/over arbitrary schemes. Ex., every adjoint group of type D_n is some $\mathbf{PGO}^+_{(\mathcal{A},\sigma,f)}$ with $\deg(\mathcal{A})=2n$.

Deformation Problems

- $X \hookrightarrow X'$ closed embedding defined by $\mathcal{J} \subset \mathcal{O}_{X'}$, $\mathcal{J}^2 = 0$.
- G' an algebraic group on X'.
- \mathcal{P} a $G'|_{X}$ —torsor (on X).

When does \mathcal{P} come from X'?

Theorem (Illusie, 1972)

- (i) $\exists \mathcal{P}' \text{ a } G'\text{-torsor on } X' \text{ such that } \mathcal{P}'|_X \cong \mathcal{P} \text{ if and only if } obs(\mathcal{P}) \in H^2(X,\mathfrak{Lie}(\mathcal{A}\text{ut}(\mathcal{P})) \otimes_{\mathcal{O}_X} \mathcal{J}) \text{ is zero.}$
- (ii) If obs(\mathcal{P}) = 0, all such \mathcal{P}' are classified by $H^1(X, \mathfrak{Lie}(Aut(\mathcal{P})) \otimes_{\mathcal{O}_X} \mathcal{J})$

(iii) For a fixed
$$\mathcal{P}'$$
,

$$\mathsf{Ker}\,(\operatorname{\mathcal{A}\!\mathit{ut}}(\mathcal{P}') \xrightarrow{\mathsf{res}} \operatorname{\mathcal{A}\!\mathit{ut}}(\mathcal{P})) \cong H^0(X, \mathfrak{Lie}(\operatorname{\mathcal{A}\!\mathit{ut}}(\mathcal{P})) \otimes_{\mathcal{O}_X} \mathcal{J}).$$

Relative Obstructions

If $\varphi \colon G' \to H'$ is a morphism of groups on X', we get

$$H^1(X, G') \to H^1(X, H')$$

 $[\mathcal{P}] \mapsto [\varphi_*(\mathcal{P})]$

as well as

$$H^2(X, \mathfrak{Lie}(\mathcal{A}ut(\mathcal{P})) \otimes_{\mathcal{O}_X} \mathcal{J}) \to H^2(X, \mathfrak{Lie}(\mathcal{A}ut(\varphi_*(\mathcal{P}))) \otimes_{\mathcal{O}_X} \mathcal{J})$$

 $\mathsf{obs}(\mathcal{P}) \mapsto \mathsf{obs}(\varphi_*(\mathcal{P})).$

Question

We consider $PGO_{2n} \hookrightarrow PGL_{2n}$.

$$H^1(X, \mathbf{PGO}_{2n}) \to H^1(X, \mathbf{PGL}_{2n})$$

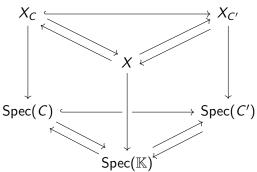
 $[(\mathcal{A}, \sigma, f)] \mapsto [\mathcal{A}].$

Is there an example where (A, σ, f) does not deform, but A does?

Convenient Setup

Let \mathbb{K} be a field, (C, \mathfrak{m}) , (C', \mathfrak{m}') Artinian local \mathbb{K} -algebras with residue field \mathbb{K} .

A surjection C' woheadrightarrow C with kernel $J \subset C'$ such that $J \cdot \mathfrak{m}' = 0$ is a *small extension*.



Tangent-Obstruction Theory

Let \mathcal{P} be a G-torsor on X.

$$\begin{array}{c} \mathcal{D} \colon \mathsf{Art}_{\mathbb{K}} \to \mathfrak{Sets} \\ \mathcal{C} \mapsto \{\mathcal{P}' \mid \mathcal{P}' \text{ is a } \mathcal{G}|_{X_{\mathcal{C}}}\text{-torsor}, \mathcal{P}'|_{X} \cong \mathcal{P}\}/\sim \end{array}$$

For any small extension $C' \rightarrow C$ with kernel J, we get

$$H^1(X, \mathfrak{Lie}(\mathcal{A}ut(\mathcal{P}))) \otimes_{\mathbb{K}} J \to D(C') \xrightarrow{\mathsf{res}} D(C) \to H^2(X, \mathfrak{Lie}(\mathcal{A}ut(\mathcal{P}))) \otimes_{\mathbb{K}} J,$$

a short exact sequence of pointed sets.

 \bullet The cohomology sets are always over X.

Tangent-Obstruction Theory

Given
$$\varphi \colon G \to H$$
, and $\mathcal P$ a G -torsor. Letting

$$\begin{array}{c} D \colon \mathsf{Art}_{\mathbb{K}} \to \mathfrak{Sets} \\ C \mapsto \{\mathsf{Deformations} \ \mathsf{of} \ \mathcal{P} \ \mathsf{to} \ X_C\}/\sim \end{array}$$

$$F\colon \mathsf{Art}_\mathbb{K} o \mathfrak{Sets}$$
 $C\mapsto \{\mathsf{Deformations\ of\ } arphi_*(\mathcal{P})\ \mathsf{to\ } X_C\}/\sim$

we have

$$H^1(X, "\mathcal{P}") \otimes_{\mathbb{K}} J \longrightarrow D(C') \rightarrow D(C) \longrightarrow H^2(X, "\mathcal{P}") \otimes_{\mathbb{K}} J$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$
 $H^1(X, "\varphi_*(\mathcal{P})") \otimes_{\mathbb{K}} J \rightarrow F(C') \rightarrow F(C) \rightarrow H^2(X, "\varphi_*(\mathcal{P})") \otimes_{\mathbb{K}} J$

Some Relatively Unobstructed Cases

If
$$2\in \mathcal{O}_X^{ imes}$$
 or if $\deg(A)=2$, then
$$(\mathcal{A},\sigma,f) \ \mathsf{deforms} \Leftrightarrow \mathcal{A} \ \mathsf{deforms}.$$

In general,

$$\mathfrak{Lie}(\mathsf{PGL}_{\mathcal{A}}) \cong \mathcal{A}/\mathcal{O}_X$$

 $\mathfrak{Lie}(\mathsf{PGO}_{(\mathcal{A},\sigma,f)}) \cong \{x \in \mathcal{A}/\mathcal{O}_X \mid x + \overline{\sigma}(x) = 0, f \circ \mathrm{ad}(x)|_{Sm_{\mathcal{A},\sigma}} = 0\}.$

If
$$2\in\mathcal{O}_X^{\times}$$
, $\mathfrak{Lie}(\mathbf{PGO}_{(\mathcal{A},\sigma,f)})\cong\mathcal{A}\ell t_{\mathcal{A},\sigma}$ "= $\{a-\sigma(a)\mid a\in\mathcal{A}\}$ ", and

$$0 \longrightarrow \mathfrak{Lie}(\mathsf{PGO}_{(\mathcal{A},\sigma,f)}) \hookrightarrow \mathfrak{Lie}(\mathsf{PGL}_{\mathcal{A}}) \longrightarrow \mathsf{N} \longrightarrow 0$$

is a splitting.

Some Relatively Unobstructed Cases

Over any scheme when deg(A) = 2,

$$\mathfrak{Lie}(\mathbf{PGL}_2) \cong \mathsf{M}_2(\mathcal{O}_X)/\mathcal{O}_X \cong \left\{ \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \right\}$$
$$\mathfrak{Lie}(\mathbf{PGO}_{(\mathsf{M}_2(\mathcal{O}_X), \sigma_2, f_2)}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \right\}.$$

So,

$$\mathfrak{Lie}(\textbf{PGL}_2) \cong \mathfrak{Lie}(\textbf{PGO}_{(M_2(\mathcal{O}_X),\sigma_2,f_2)}) \oplus \textit{N}.$$

and this splitting is stabilized by PGO_2 , so it twists to

$$\mathfrak{Lie}(\mathsf{PGL}_{\mathcal{A}}) \cong \mathfrak{Lie}(\mathsf{PGO}_{(\mathcal{A},\sigma,f)}) \oplus \mathsf{N}'.$$

A Relatively Obstructed Example

The next case is $\deg(\mathcal{A})=4$ and $2\notin\mathcal{O}_X^{\times}.$ Here, we construct an example where

 (A, σ, f) does not deform, but A deforms.

Ingredients:

- Norm equivalence $A_1^2 \stackrel{\sim}{\to} D_2$.
- Igusa surface X over \mathbb{K} , char(\mathbb{K}) = 2.
- The example is a biquaternion algebra on $X_{k[x]/\langle x^2\rangle}$ which does not deform to $X_{k[x]/\langle x^3\rangle}$.

Norm Equivalence

Joint work with Philippe Gille and Erhard Neher. Two stacks:

$$egin{aligned} A_1^2 &
ightarrow \mathfrak{Sch}_X & D_2 &
ightarrow \mathfrak{Sch}_X \ (T' &
ightarrow T, \mathcal{Q}) &
ightarrow T & (T, (\mathcal{A}, \sigma, f)) &
ightarrow T \end{aligned}$$

- (i) $T' \rightarrow T$ degree 2 étale cover
- (i) (σ, f) quadratic pair

(ii) Q is on T', deg(Q) = 2

(ii) deg(A) = 4.

We have an equivalence of stacks

$$N \colon A_1^2 \stackrel{\sim}{\to} D_2.$$

$$(T \sqcup T \to T, (\mathcal{Q}_1, \mathcal{Q}_2)) \mapsto (T, (\mathcal{Q}_1 \otimes_{\mathcal{O}_T} \mathcal{Q}_2, \sigma_1 \otimes \sigma_2, f_{\otimes})).$$

Norm Equivalence

Let $X \hookrightarrow X'$ be an infinitesimal thickening.

$$egin{aligned} egin{aligned} (\mathcal{T}
ightarrow \mathcal{X}', \mathcal{Q}') & \longmapsto & N \ & & \downarrow^{\mathrm{res}} & \downarrow^{\mathrm{res}} \ & & \downarrow^{\mathrm{res}} \ & & \downarrow^{X} & & \downarrow^{X} \ & & \downarrow^$$

- Étale extensions have trivial deformation theory. $\Rightarrow T = X' \sqcup X'$.
- $\bullet \ \Rightarrow \mathcal{Q}' = \big(\mathcal{Q}_1', \mathcal{Q}_2'\big).$

$$(\mathcal{Q}_1 \otimes_{\mathcal{O}_X} \mathcal{Q}_2, \sigma_1 \otimes \sigma_2, f_{\otimes})$$
 deforms to X' \uparrow

$$\downarrow$$

Both Q_1 and Q_2 deform to X'.

Igusa Surface

Let E be an ordinary elliptic curve over \mathbb{K} , $\mathbb{K} = \overline{\mathbb{K}}$, $\operatorname{char}(\mathbb{K}) = 2$.

$$E[2] \cong \mu_2 \times \mathbb{Z}/2\mathbb{Z}$$
 and $E[2](\mathbb{K}) = \{0, t\}.$

 $\mathbb{Z}/2\mathbb{Z}$ acts on $E \times_{\mathbb{K}} E$ by

$$(a,b)\mapsto (-a,b+t)$$

This action has a smooth, projective quotient

$$\pi \colon E \times_{\mathbb{K}} E \to X$$

X is an Igusa surface.

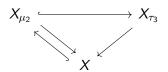
- It has trivial canonical bundle, $K_X \cong \mathcal{O}_X$.
- $\dim_{\mathbb{K}}(H^1(X, \mathcal{O}_X)) = 2$, $\dim_{\mathbb{K}}(H^2(X, \mathcal{O}_X)) = 1$.

Igusa Surface

 The Picard scheme of X is non-reduced (Igusa, Jensen, Srinivas+Mehta).

$$\operatorname{Pic}_{X/\mathbb{K}}^0 \cong \mu_2 \times E/\langle t \rangle.$$

 $\mu_2 = \operatorname{Spec}(k[x]/\langle x^2 \rangle)$, and let $\tau_3 = \operatorname{Spec}(k[x]/\langle x^3 \rangle)$. Consider



For ${\mathcal L}$ on X, ${\mathcal L}=(x,y)\in {\sf Pic}^0_{X/{\mathbb K}}$,

$$T_{\mathcal{L}}(\mathsf{Pic}^0_{X/\mathbb{K}}) \cong T_{x}(\mu_2) \oplus T_{y}(E/\langle t \rangle) \cong H^1(X, \mathcal{O}_X).$$

any \mathcal{L}' with non-trivial $T_{\scriptscriptstyle \mathcal{X}}(\mu_2)$ component does not deform to $X_{\! au_3}.$

Sketch of the Construction

Using explicit cocycles, define

$$\mathcal{Q} = \mathcal{E}$$
nd $_{\mathcal{O}_X}(\mathcal{V})$ and $\mathcal{P} = \mathcal{E}$ nd $_{\mathcal{O}_X}(\mathcal{W})$

two quaternions on X. Calculate: $\mathcal{Q}(X) = \mathcal{P}(X) = (\mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{P})(X) = \mathbb{K}$.

Proposition

Let $\operatorname{char}(\mathbb{K})=2$ and X a smooth, projective surface over \mathbb{K} with trivial canonical bundle. Let (\mathcal{A},σ,f) be an Azumaya algebra with quadratic pair on X such that $\mathcal{A}(X) \xrightarrow{\operatorname{Id} + \sigma} \mathcal{A}\ell t_{\mathcal{A},\sigma}(X)$ and $\operatorname{Sym}_{\mathcal{A},\sigma}(X) \xrightarrow{f} \mathcal{O}_X(X)$ are both zero. Then,

$$H^2(X, \mathfrak{Lie}(\mathsf{PGO}_{(A,\sigma,f)})) \to H^2(X, \mathfrak{Lie}(\mathsf{PGL}_A))$$

is zero also.

- $(Q \otimes_{\mathcal{O}_X} \mathcal{P}, \sigma_{\mathcal{O}} \otimes \sigma_{\mathcal{P}}, f_{\otimes})$ satisfies the proposition.
- \Rightarrow any deformation of $\mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{P}$ to X_{μ_2} itself deforms to X_{τ_3} .

Sketch of the Construction

For $\mathcal Q$ (likewise for $\mathcal P$)

$$0 \to \mathcal{O}_X \to \mathcal{Q} \to \mathcal{Q}/\mathcal{O}_X \to 0$$

induces

Serre dual to

$$0 o (\mathcal{Q}/\mathcal{O}_{X})^{ee}(X) o \mathcal{Q}^{ee}(X) \overset{0}{ o} \mathcal{O}_{X}^{ee}(X)$$

- $\Rightarrow H^2(X, \mathcal{Q}) \stackrel{\sim}{\to} H^2(X, \mathcal{Q}/\mathcal{O}_X)$
- ullet \Rightarrow $\mathcal{Q} = \operatorname{\it End}_{\mathcal{O}_X}(\mathcal{V})$ deforms if and only if \mathcal{V} deforms.

Sketch of the Construction

 $\mathsf{Trd}_{\mathcal{Q}} \colon \mathcal{Q} \to \mathcal{O}_X$ and $\mathsf{Trd}_{\mathcal{P}} \colon \mathcal{P} \to \mathcal{O}_X$ induce

$$H^1(X,\mathcal{Q}) o H^1(X,\mathcal{O}_X) \qquad \qquad H^1(X,\mathcal{P}) o H^1(X,\mathcal{O}_X) \ [\mathcal{V}'] \mapsto [\det(\mathcal{V}')] \qquad \qquad [\mathcal{W}'] \mapsto [\det(\mathcal{W}')].$$

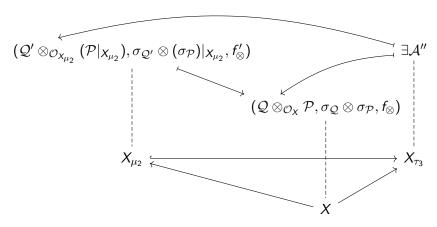
By analyzing the module structures of \mathcal{Q}, \mathcal{P} , see that these maps have linearly independent images in

$$H^1(X, \mathcal{O}_X) = T_x(\mu_2) \oplus T_y(E/\langle t \rangle)$$

- ullet \Rightarrow some \mathcal{V}' or \mathcal{W}' on X_{μ_2} has obstructed determinant bundle.
- \Rightarrow some \mathcal{V}' or \mathcal{W}' is obstructed
- \Rightarrow some \mathcal{Q}' or \mathcal{P}' is obstructed.

The Example

• Take \mathcal{Q}' on X_{μ_2} deforming \mathcal{Q} , obstructed on X_{τ_3} .



• No deformation (A'', σ'', f'') on X_{τ_3} exists.

Thank You