# Chevalley Generators in Étale Stalks

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# Chevalley-Demazure Group Schemes

If $\Phi$ is a reduced irreducible root system, $L_r \subseteq L \subseteq L_w$ a lattice, then there exists a unique functor

$$\mathbf{G}_L(\Phi, \_\_)\colon \mathbf{Alg}_{\mathbb{Z}} \to \mathbf{Grp}$$

which recovers the classical semisimple linear algebraic group over algebraically closed fields.

**Example**

$$\mathbf{SL}_n\colon R \mapsto \{x \in \mathrm{M}_n(R) \mid \det(x) = 1\}$$

$$\mathbf{PGL}_n\colon R \mapsto \mathrm{Aut}_R(\mathrm{M}_n(R))$$

$$\mathbf{SO}_n\colon R \mapsto \{x \in \mathrm{M}_n(R) \mid \det(x) = 1, x\Omega x^T\Omega = I\}, \Omega = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

$$\mathbf{Sp}_{2n}\colon R \mapsto \{x \in \mathrm{M}_{2n}(R) \mid -x\Psi x^T\Psi = I\}, \Psi = \begin{bmatrix} & \Omega \\ -\Omega & \end{bmatrix}$$

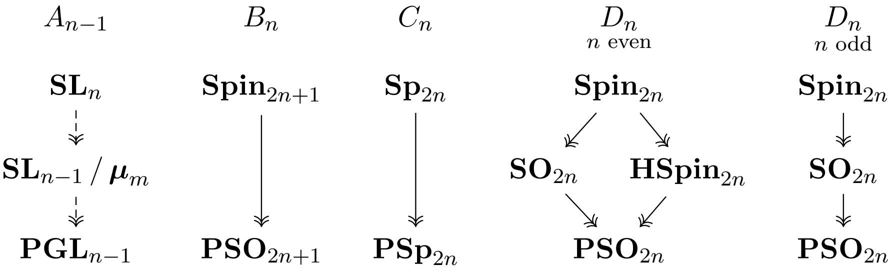
# Linear Algebraic Groups over a Field

A linear algebraic group over $\mathbb{F}$ is a functor

$$\begin{aligned} \mathbf{G}\colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\ R &\mapsto \mathrm{Hom}_{\mathbb{F}}(H, R) \end{aligned}$$

represented by a finitely presented $\mathbb{F}$-Hopf algebra $H$.

- When $\mathbf{G}$ is of type $A_{n-1}$ assume $\mathrm{char}(\mathbb{F}) \nmid n$,
- When $\mathbf{G}$ is of type $B, C$, or $D$ assume $\mathrm{char}(\mathbb{F}) \neq 2$.



# Chevalley Generators

Let $\mathbf{G}$ be a group scheme with root system $\Phi$. For each $\alpha \in \Phi$, there exists

$$x_\alpha \colon \mathbb{G}_a \to \mathbf{G}$$

such that, for $R \in \mathbf{Alg}_{\mathbb{F}}$

- $x_\alpha(t)x_\alpha(u) = x_\alpha(t+u)$ for $t, u \in R$,
- $h_\alpha(t)h_\alpha(u) = h_\alpha(tu)$ for $t, u \in R^\times$,
- $(x_\alpha(t), x_\beta(u)) = \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in\Phi}} x_{i\alpha+j\beta}(c_{ij}t^i u^j)$.

where $h_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)x_\alpha(-1)x_{-\alpha}(1)x_\alpha(-1)$.

The *Elementary subgroup* over $R$ is

$$E_{\mathbf{G}}(R) = \langle x_\alpha(t) \mid \alpha \in \Phi, t \in R \rangle \leq \mathbf{G}(R).$$

# Known Results

$E_{\mathbf{G}}(R) = \mathbf{G}(R)$ if,

- $R = \mathbb{K} = \overline{\mathbb{K}}$ is an algebraically closed field
- $\mathbf{G}$ is s.c. and $R = \mathbb{K}$ a field
- $\mathbf{G}$ is s.c. and $R$ is a semi-local ring [Matsumoto '66]
- $\mathbf{G}$ is s.c. and $R$ is a Euclidean ring [Steinberg '68]
- $\mathbf{G}$ is s.c. and $R$ is a Hasse domain [Bass, Milnor, Serre '67] [Matsumoto '66]
- $\mathbf{G}$ is s.c. and $R = \mathbb{K}[x_1, \ldots, x_n]$ [Suslin '77]

## Example

*Consider* $\mathbf{SO}_5(\mathbb{Q})$. $\Phi = \{\pm e_1 \pm e_2, \pm e_1, \pm e_2\}$.

$$x_{e_1-e_2}(t) = \begin{bmatrix} 1 & t & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & -t \\ & & & & 1 \end{bmatrix} \qquad x_{e_1+e_2}(t) = \begin{bmatrix} 1 & & & t & \\ & 1 & & & -t \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

$$x_{-e_1+e_2}(t) = \begin{bmatrix} 1 & & & & \\ t & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & -t & 1 \end{bmatrix} \qquad x_{-e_1-e_2}(t) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ t & & 1 & & \\ & & & 1 & \\ & -t & & & 1 \end{bmatrix}$$

$$x_{e_1}(t) = \begin{bmatrix} 1 & & 2t & & -t^2 \\ & 1 & & & \\ & & 1 & & -t \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \qquad x_{-e_1}(t) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ t & & 1 & & \\ & & & 1 & \\ -t^2 & & -2t & & 1 \end{bmatrix}$$

$$x_{e_2}(t) = \begin{bmatrix} 1 & & & & \\ & 1 & 2t & -t^2 & \\ & & 1 & -t & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \qquad x_{-e_2}(t) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & t & 1 & & \\ & -t^2 & -2t & 1 & \\ & & & & 1 \end{bmatrix}$$

## Example

$$h_{e_1-e_2}(t) = \begin{bmatrix} t & & & & \\ & t^{-1} & & & \\ & & 1 & & \\ & & & t & \\ & & & & t^{-1} \end{bmatrix} \quad h_{e_2}(t) = \begin{bmatrix} 1 & & & & \\ & t^2 & & & \\ & & 1 & & \\ & & & t^{-2} & \\ & & & & 1 \end{bmatrix}$$

*Since* $\sqrt{2} \notin \mathbb{Q}$*, we have*

$$\begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & \frac{1}{2} & \\ & & & & 1 \end{bmatrix} \in \mathbf{SO}_5(\mathbb{Q}),$$

*but not in* $E_{\mathbf{SO}_5}(\mathbb{Q})$.

# Sites and Sheaves

A site is a category $\mathcal{C}$ together with a collection of coverings $\mathrm{Cov}(\mathcal{C})$ consisting of families of morphism $\{U_i \to U\}_{i \in I}$ such that

1. If $V \xrightarrow{\sim} U$, then $\{V \xrightarrow{\sim} U\} \in \mathrm{Cov}(\mathcal{C})$.
2. If $\{U_i \to U\}_{i \in I}, \{U_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{U_{ij} \to U\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{C})$.
3. If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$, then $\{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$.

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A contravariant functor $\mathcal{F} \colon \mathcal{C} \to \mathbf{Sets}$ is a *presheaf*. $\mathcal{F}$ is a *sheaf* if for all $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$,

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer diagram.

# The Étale Site

Let $\mathbf{Aff}_{\mathbb{F}}$ be the category of affine schemes over $\operatorname{Spec}(\mathbb{F})$.

$$\begin{aligned} \mathbf{Aff}_{\mathbb{F}} &\cong \mathbf{Alg}_{\mathbb{F}} \\ S &\mapsto \mathcal{O}_S(S) \\ \operatorname{Spec}(R) &\hookleftarrow R \end{aligned}$$

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$$\begin{aligned} \mathbf{G} \colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\ R &\mapsto \operatorname{Hom}_{\mathbb{F}}(H, R) \\ &\updownarrow \\ \hat{\mathbf{G}} \colon \mathbf{Aff}_{\mathbb{F}} &\to \mathbf{Sets} \\ U &\mapsto \operatorname{Hom}_{\mathbf{Aff}_{\mathbb{F}}}(U, \operatorname{Spec}(H)) \end{aligned}$$

# The Étale Site

The *étale site* is $\mathbf{Aff}_{\mathbb{F}}$ equipped with covers

$$\{U_i \to U\}_{i=1}^{n}$$

with each $U_i \to U$ étale.
**Fact:** The étale site is *subcanonical.* So

$$\begin{aligned} \hat{\mathbf{G}} \colon \mathbf{Aff}_{\mathbb{F}} &\to \mathbf{Sets} \\ U &\mapsto \mathrm{Hom}_{\mathbf{Aff}_{\mathbb{F}}}(U, \mathrm{Spec}(H)) \end{aligned}$$

are sheaves.

# Points and Stalks

**Definition**

*Let $X \in \mathbf{Aff}_{\mathbb{F}}$. A geometric point of $\mathbf{Aff}_{\mathbb{F}}$ over $X$ is a map*

$$p\colon \operatorname{Spec}(\mathbb{K}) \to X$$

*where $\mathbb{K} \in \mathbf{Alg}_{\mathbb{F}}$ is an algebraically closed field.*

Étale neighbourhood:

$$\operatorname{Spec}(\mathbb{K}) \longrightarrow U \xrightarrow{\varphi} X, \qquad p\colon \operatorname{Spec}(\mathbb{K}) \to X$$

Morphism of neighbourhoods:

$$\begin{array}{ccccc} & & U_1 & & \\ & \nearrow & \downarrow & \searrow & \\ \operatorname{Spec}(\mathbb{K}) & & & & X \\ & \searrow & \downarrow & \nearrow & \\ & & U_2 & & \end{array}$$

# Points and Stalks

**Definition**

*If $\mathcal{F}$ is a presheaf on $\mathbf{Aff}_{\mathbb{F}}$, $p\colon \operatorname{Spec}(\mathbb{K}) \to X$ a geometric point, the stalk of $\mathcal{F}$ at $p$ is*

$$\mathcal{F}_p = \operatorname*{colim}_{U \to X} \mathcal{F}(U)$$

**Fact:** $\mathbf{Aff}_{\mathbb{F}}$ has enough geometric points.

$$\mathcal{F} \to \mathcal{G} \text{ is injective (resp. surjective)} \Leftrightarrow \mathcal{F}_p \to \mathcal{G}_p \text{ is inj. (resp. surj)} \forall p$$

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Let $\mathbf{G}_1 \to \mathbf{G}_2$ be a morphism of group schemes.

Injective as étale sheaves $\Leftrightarrow$ Injective as algebraic groups

Surjective as étale sheaves $\Rightarrow$ Surjective as algebraic groups

# What are these stalks?

Consider the sheaf

$$\mathcal{O} \colon \mathbf{Aff}_{\mathbb{F}} \to \mathbf{Rings}$$
$$\operatorname{Spec}(R) \mapsto R.$$

If $p \colon \operatorname{Spec}(K) \to \operatorname{Spec}(R)$ is a geometric points, $\operatorname{Img}(p) = \mathfrak{p}$,

$$\mathcal{O}_p = (R_{\mathfrak{p}})^{\mathrm{sh}}$$

**Definition**

*A local ring $R$ is called local strictly Henselian if*

1. *$R/\mathfrak{M}$ is separably close, and*
2. *If $f(x) \in R[x]$ is a polynomial whose image in $R/\mathfrak{M}[x]$ factors, then that factorization lifts to one in $R[x]$.*

# Stalks of Group Schemes

Let $\mathbf{G}$ be a group scheme, $R \in \mathbf{Alg}_{\mathbb{F}}$, and $p\colon \operatorname{Spec}(K) \to \operatorname{Spec}(R)$ a geometric point over $\mathfrak{p}$. Then

$$\hat{\mathbf{G}}_p = \mathbf{G}((R_{\mathfrak{p}})^{\mathrm{sh}})$$

**Proof.**

$$\hat{\mathbf{G}}_p = \operatorname*{colim}_{U \to X} \hat{\mathbf{G}}(U) = \operatorname*{colim}_{(S,\mathfrak{q},\alpha)} \mathbf{G}(S) = \operatorname*{colim}_{(S,\mathfrak{q},\alpha)} \operatorname{Hom}_{\mathbb{F}}(H, S)$$
$$= \operatorname{Hom}_{\mathbb{F}}(H, \operatorname*{colim}_{(S,\mathfrak{q},\alpha)} S) = \operatorname{Hom}_{\mathbb{F}}(H, (R_{\mathfrak{p}})^{\mathrm{sh}}) = \mathbf{G}((R_{\mathfrak{p}})^{\mathrm{sh}}).$$

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**Proposition**

$$\mathbf{G}((R_{\mathfrak{p}})^{\mathrm{sh}}) = E_{\mathbf{G}}((R_{\mathfrak{p}})^{\mathrm{sh}}).$$

Thank You