

Cohomological Obstructions to Quadratic Pairs over Schemes

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$\text{char}(\mathbb{F}) \neq 2$

{quadratic forms} \longleftrightarrow {Symmetric bilinear forms}

$$(q: V \rightarrow \mathbb{F}) \longmapsto b_q(x, y) = q(x + y) - q(x) - q(y)$$

$$q_b(x) = b(x, x) \longleftarrow (b: V \times V \rightarrow \mathbb{F})$$

- $q = \frac{1}{2}q_{b_q}$ and $b = \frac{1}{2}b_{q_b}$.
- $\text{PGO}^+(V, q) = \text{PGO}^+(\text{End}_{\mathbb{F}}(V), \sigma_q)$ when q is regular.
- $\text{PGO}^+(A, \sigma)$ are smooth of type D .

Involutions

In any characteristic, let (A, σ) be a c.s.a. with \mathbb{F} -linear involution.

- σ is *orthogonal* if σ_{sep} on $A_{\text{sep}} = M_r(\mathbb{F}_{\text{sep}})$ is adjoint to a symmetric bilinear form.
- σ is *weakly-symplectic* if σ_{sep} is adjoint to a skew-symmetric bilinear form.
- σ is *symplectic* if σ_{sep} is adjoint to an alternating bilinear form.

If $\text{char}(\mathbb{F}) = 2$, $O(A, \sigma)$ may not be smooth.

Quadratic Pairs

Definition

Let A be a central simple \mathbb{F} -algebra. A *quadratic pair* on A is (A, σ, f) where

- σ is an orthogonal involution on A , and
- $f: \text{Sym}(A, \sigma) \rightarrow \mathbb{F}$ is a linear map satisfying

$$f(a + \sigma(a)) = \text{Trd}_A(a)$$

for all $a \in A$. Here $\text{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\}$.

- If $\text{char}(\mathbb{F}) \neq 2$, $\Rightarrow f = \frac{1}{2} \text{Trd}_A$.
- The orthogonal group

$$\text{O}(A, \sigma, f) = \{a \in A \mid \sigma(a) = a^{-1}, f(asa^{-1}) = f(s)\}$$

is smooth. $\text{O}^+(A, \sigma, f)$ is semisimple type D .

Quadratic Pairs and Quadratic Forms

- If $A = \text{End}_{\mathbb{F}}(V)$, then σ is adjoint to a regular $b: V \times V \rightarrow \mathbb{F}$.

$$(V \otimes_{\mathbb{F}} V, \text{switch}) \xrightarrow{\sim} (\text{End}_{\mathbb{F}}(V), \sigma)$$
$$x \otimes y \mapsto b(x, _)y$$

Theorem (KMRT)

$$\{\text{quadratic pairs involving } \sigma\} \leftrightarrow \{q \text{ whose polar is } b\}$$

Idea: $f(x \otimes x) = q(x)$ and $f(x \otimes y + y \otimes x) = b(x, y)$

Classification

Let (A, σ) c.s.a. with orthogonal involution. Define

$$\text{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\}$$

$$\text{Skew}(A, \sigma) = \{a \in A \mid \sigma(a) = -a\}$$

$$\text{Symd}(A, \sigma) = \{a + \sigma(a) \mid a \in A\}$$

$$\text{Alt}(A, \sigma) = \{a - \sigma(a) \mid a \in A\}$$

- The trace form

$$A \times A \mapsto \mathbb{F}$$

$$(a, b) \mapsto \text{Trd}_A(ab)$$

is a regular, symmetric, bilinear form.

- $\text{Sym}(A, \sigma)^\perp = \text{Alt}(A, \sigma)$ and $\text{Alt}(A, \sigma)^\perp = \text{Sym}(A, \sigma)$

Classification

Theorem (KMRT)

If (A, σ, f) is a quadratic pair on A , then there exists $\ell \in A$ such that

- $\ell + \sigma(\ell) = 1$,
- $f(s) = \text{Trd}_A(\ell s)$,
- ℓ is unique up to addition by an element from $\text{Alt}(A, \sigma)$.

Conversely, for any $\ell \in A$ satisfying $\ell + \sigma(\ell) = 1$, the linear map

$$\begin{aligned} f: \text{Sym}(A, \sigma) &\rightarrow \mathbb{F} \\ s &\mapsto \text{Trd}_A(\ell s) \end{aligned}$$

makes (A, σ, f) a quadratic pair. This form only depends on $[\ell] \in A/\text{Alt}(A, \sigma)$.

Over a Scheme

Generalized by Calmès and Fasel.

- S is a fixed base scheme
- \mathfrak{Sch}_S site with the fppf topology
- \mathcal{O} the sheaf of rings $\mathcal{O}(T) = \Gamma(T, \mathcal{O}_T)$ for $T \in \mathfrak{Sch}_S$
- \mathcal{A} an Azumaya \mathcal{O} -algebra of constant degree. So $\exists \{T_i \rightarrow S\}_{i \in I}$ such that

$$\mathcal{A}|_{T_i} \cong M_r(\mathcal{O})|_{T_i}$$

Definition

Let \mathcal{A} be an Azumaya \mathcal{O} -algebra. A *quadratic pair* on \mathcal{A} is (\mathcal{A}, σ, f) where

- σ is an orthogonal involution on \mathcal{A} , and
- $f: \mathit{Sym}_{\mathcal{A}, \sigma} \rightarrow \mathcal{O}$ is an \mathcal{O} -linear natural transformation satisfying

$$f(a + \sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(a)$$

for all $T \in \mathfrak{Sch}_S$ and $a \in \mathcal{A}(T)$.

Question

- Given (\mathcal{A}, σ) , when can it be extended to (\mathcal{A}, σ, f) ?
 - Given (\mathcal{A}, σ) , what is a classification of all possible (\mathcal{A}, σ, f) ?
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$\text{Id} + \sigma: \mathcal{A} \rightarrow \mathcal{A}$ and $\text{Id} - \sigma: \mathcal{A} \rightarrow \mathcal{A}$. Define

- $\text{Sym}_{\mathcal{A}, \sigma} = \ker(\text{Id} - \sigma)$
- $\text{Skew}_{\mathcal{A}, \sigma} = \ker(\text{Id} + \sigma)$
- $\text{Symd}_{\mathcal{A}, \sigma} = \text{Im}(\text{Id} + \sigma)$
- $\text{Altd}_{\mathcal{A}, \sigma} = \text{Im}(\text{Id} - \sigma)$

Over an Affine Scheme

If S is an affine scheme,

- $\text{Sym}_{\mathcal{A},\sigma}, \text{Alt}_{\mathcal{A},\sigma}$ are direct summands of \mathcal{A} when σ is orthogonal, and mutually perpendicular w.r.t. the trace form.
 - So, linear forms $f: \text{Sym}_{\mathcal{A},\sigma} \rightarrow \mathcal{O}$ correspond to $\ell \in \mathcal{A}(S)$ with $\ell + \sigma(\ell) = 1$ up to addition by an element of $\text{Alt}(\mathcal{A}, \sigma)$.
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If S is any scheme, and (\mathcal{A}, σ, f) a quadratic pair

- $\{U_i \rightarrow S\}_{i \in I}$ an affine open cover
- $(\mathcal{A}|_{U_i}, \sigma|_{U_i}, f|_{U_i})$ will be given by some $\ell_i \in \mathcal{A}(U_i)$ with $\ell_i + \sigma(\ell_i) = 1$.
- $\Rightarrow 1 \in \text{Symd}_{\mathcal{A},\sigma}(U_i)$ for all $i \in I$,
- $\Rightarrow 1 \in \text{Symd}_{\mathcal{A},\sigma}(S)$.

Locally Quadratic Involutions

Definition

We call (\mathcal{A}, σ) an Azumaya \mathcal{O} -algebra with *locally quadratic involution* if $1 \in \text{Symd}_{\mathcal{A}, \sigma}(S)$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Skew}_{\mathcal{A}, \sigma} & \hookrightarrow & \mathcal{A} & \xrightarrow{\text{Id} + \sigma} & \text{Symd}_{\mathcal{A}, \sigma} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \text{Skew}_{\mathcal{A}, \sigma} / \text{Alt}_{\mathcal{A}, \sigma} & \hookrightarrow & \mathcal{A} / \text{Alt}_{\mathcal{A}, \sigma} & \xrightarrow{\pi} & \text{Symd}_{\mathcal{A}, \sigma} \longrightarrow 0 \end{array}$$

Cohomological Obstructions

$$\begin{array}{ccccccc}
 \mathcal{A}(S) & \longrightarrow & \text{Symd}_{\mathcal{A},\sigma}(S) & \longrightarrow & H^1(S, \text{Skew}_{\mathcal{A},\sigma}) & \longrightarrow & \dots \\
 \downarrow & & \parallel & & \downarrow & & \\
 (\mathcal{A}/\text{Alt}_{\mathcal{A},\sigma})(S) & \xrightarrow{\pi(S)} & \text{Symd}_{\mathcal{A},\sigma}(S) & \longrightarrow & H^1(S, \text{Skew}_{\mathcal{A},\sigma}/\text{Alt}_{\mathcal{A},\sigma}) & \longrightarrow & \dots
 \end{array}$$

$$\begin{array}{ccc}
 1 & \longrightarrow & \Omega(\mathcal{A}, \sigma) \\
 \parallel & & \downarrow \\
 1 & \longrightarrow & \omega(\mathcal{A}, \sigma)
 \end{array}$$

We call $\Omega(\mathcal{A}, \sigma)$ the *strong obstruction* and $\omega(\mathcal{A}, \sigma)$ the *weak obstruction*.

Cohomological Obstructions

Theorem (Gille, Neher, R.)

- $\Omega(\mathcal{A}, \sigma) = 0 \Leftrightarrow \exists f = \text{Trd}_{\mathcal{A}}(\ell_{_})$ with $\ell + \sigma(\ell) = 1$ for $\ell \in \mathcal{A}(S)$.
- $\omega(\mathcal{A}, \sigma) = 0 \Leftrightarrow \exists f: \text{Sym}_{\mathcal{A}, \sigma} \rightarrow \mathcal{O}$ making (\mathcal{A}, σ, f) a quadratic pair.
- *There is a classification*

$$\{f \text{ extending } (\mathcal{A}, \sigma)\} \leftrightarrow \pi(S)^{-1}(1) \subset (\mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma})(S).$$

Proof.

- Any (\mathcal{A}, σ, f) is given locally by ℓ_i , which glue to a section $\lambda \in (\mathcal{A}/\text{Alt}_{\mathcal{A}, \sigma})(S)$.
- Any global section λ is locally given by ℓ_i from \mathcal{A} , the $f_i = \text{Trd}_{\mathcal{A}}(\ell_i_{_})$ glue into a global f .
- $\omega(\mathcal{A}, \sigma) = [\pi^{-1}(1)] \in H^1(S, \text{Skew}_{\mathcal{A}, \sigma}/\text{Alt}_{\mathcal{A}, \sigma})$ where

$$\pi^{-1}(1): \mathfrak{Sch}_S \rightarrow \mathfrak{Sets}, \quad \pi(T)^{-1}(1|_T)$$

An Example

Let $\text{char}(\mathbb{F}) = 2$. Take E an ordinary elliptic curve as the base scheme.

$$E \xrightarrow{\cdot 2} E$$

is an $E[2] = \mu_2 \times_{\mathbb{F}} \mathbb{Z}/2\mathbb{Z}$ torsor.

$$\mu_2 \times_{\mathbb{F}} \mathbb{Z}/2\mathbb{Z} \hookrightarrow \text{PGL}_2$$

$\Rightarrow E \wedge^{\mu_2 \times_{\mathbb{F}} \mathbb{Z}/2\mathbb{Z}} \text{PGL}_2$ is a PGL_2 -torsor, so defines \mathcal{Q} a quaternion algebra. It has canonical involution (\mathcal{Q}, σ) .

- σ is symplectic, hence also orthogonal.
- σ can be extended to a quadratic pair (\Rightarrow locally quadratic).
 $\Rightarrow \omega(\mathcal{Q}, \sigma) \neq 0$.
- $\mathcal{Q}(E) = \mathbb{F}$. So $\ell + \sigma(\ell) = 2\ell = 0$ for all $\ell \in \mathcal{Q}(E)$. $\Rightarrow \Omega(\mathcal{Q}, \sigma) = 0$.

Another Example

$\text{char}(\mathbb{F}) = 2$, $\mathbb{F} = \overline{\mathbb{F}}$. Let $\Gamma = \text{PGL}(\mathbb{F}_4)$ as an abstract group, $\Gamma_{\text{Spec}(\mathbb{F})}$ the constant group scheme.

Serre: $\exists Y \rightarrow S$ a Γ -cover between smooth projective \mathbb{F} -varieties. This is a $\Gamma_{\text{Spec}(\mathbb{F})}$ -torsor.

$$\Gamma_{\text{Spec}(\mathbb{F})} \hookrightarrow \text{PGL}_2$$

$\Rightarrow Y \wedge^{\Gamma_{\text{Spec}(\mathbb{F})}} \text{PGL}_2$ is a PGL_2 -torsor, so defines (Q, σ) a quaternion algebra with symplectic/orthogonal involution.

- σ is locally quadratic.
- (Q, σ) splits over Y .
- If we had (Q, σ, f) , then $f|_Y: \text{Sym}_{M_2(\mathcal{O})|_Y, \sigma|_Y} \rightarrow \mathcal{O}|_Y$ must be Γ -equivariant.
- $\Rightarrow f = 0$, contradiction.
- So $\omega(Q, \sigma) \neq 0$. ($\Rightarrow \Omega(Q, \sigma) \neq 0$.)

Thank You