Extending Tensor Product Maps to Non-split Linear Algebraic Groups

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Linear Algebraic Groups

Let $\mathbb{F}$ be a field, $\mathrm{char}(\mathbb{F}) \neq 2$. A *linear algebraic group* is a covariant functor

$$\begin{aligned} \mathbf{G} \colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\ R &\mapsto \mathrm{Hom}_{\mathbb{F}}(H, R) \end{aligned}$$

where H is a finitely presented $\mathbb{F}$-Hopf algebra.
A homomorphisms of linear algebraic groups are natural transformations

$$\begin{aligned} \varphi \colon \ \mathbf{G}_1 &\to \mathbf{G}_2 \\ \varphi(R) \colon \ \mathbf{G}_1(R) &\to \mathbf{G}_2(R). \end{aligned}$$

Examples

$$\Omega_n = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, \Psi_{2n} = \begin{bmatrix} & \Omega_n \\ -\Omega_n & \end{bmatrix}$$

Split
$\mathbf{SL}_n(R) = \{B \in \mathrm{M}_n(R) \mid \det(B) = 1\}$
$\mathbf{SO}_n(R) = \{B \in \mathrm{M}_n(R) \mid B\Omega_n B^T \Omega_n = I, \det(B) = 1\}$
$\mathbf{Sp}_{2n}(R) = \{B \in \mathrm{M}_n(R) \mid B\Psi_{2n} B^T \Psi_{2n} = I\}$

Non-split
$\mathbf{SL}(A)(R) = \{a \in A \otimes_{\mathbb{F}} R \mid \mathrm{Nrd}(a) = 1\}$
$\mathbf{SO}(A, \tau)(R) = \{a \in A \otimes_{\mathbb{F}} R \mid a \cdot \tau(a) = 1, \mathrm{Nrd}(a) = 1\}$
$\mathbf{Sp}(A, \psi)(R) = \{a \in A \otimes_{\mathbb{F}} R \mid a \cdot \psi(a) = 1\}$

A is a central simple $\mathbb{F}$-algebra, τ an orthogonal involution, ψ a symplectic involution.

Split Types B, C, and D

B_n

$$\mathbf{Spin}_{2n+1} \twoheadrightarrow \mathbf{SO}_{2n+1} = \mathbf{PSO}_{2n+1}$$

C_n

$$\mathbf{Sp}_{2n} \twoheadrightarrow \mathbf{PSp}_{2n}$$

D_n, n even

$$\mathbf{Spin}_{2n} \twoheadrightarrow \mathbf{SO}_{2n} \twoheadrightarrow \mathbf{PSO}_{2n}$$
$$\mathbf{Spin}_{2n} \twoheadrightarrow \mathbf{HSpin}_{2n} \twoheadrightarrow \mathbf{PSO}_{2n}$$

D_n, n odd

$$\mathbf{Spin}_{2n} \twoheadrightarrow \mathbf{SO}_{2n} \twoheadrightarrow \mathbf{PSO}_{2n}$$

n even: $Z(\mathbf{Spin}_{2n}) \cong \mu_2 \times \mu_2$. n odd: $Z(\mathbf{Spin}_{2n}) \cong \mu_4$.

Non-split Types B, C, and D

B_n

$$\mathbf{Spin}(A,\tau) \twoheadrightarrow \mathbf{SO}(A,\tau) = \mathbf{PSO}(A,\tau)$$

C_n

$$\mathbf{Sp}(A,\psi) \twoheadrightarrow \mathbf{PSp}(A,\psi)$$

D_n
n even

$$\begin{array}{ccccc} & & \mathbf{Spin}(A,\tau) & & \\ & \swarrow & & \searrow & \\ \mathbf{SO}(A,\tau) & & & & \mathbf{HSpin}(A,\tau) \\ & \searrow & & \swarrow & \\ & & \mathbf{PSO}(A,\tau) & & \end{array}$$

D_n
n odd

$$\mathbf{Spin}(A,\tau) \twoheadrightarrow \mathbf{SO}(A,\tau) \twoheadrightarrow \mathbf{PSO}(A,\tau)$$

Galois Cohomology and Central Simple Algebra

Let $\Gamma = \mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$.
We use the Galois cohomology sets $H^1(\mathbb{F}, \mathbf{G}) := H^1(\Gamma, \mathbf{G}(\mathbb{F}_{\mathrm{sep}}))$.
If A is an $\mathbb{F}$-c.s.a of degree n, then $A \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \cong \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$.

$$H^1(\mathbb{F}, \mathbf{PGL}_n) \leftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \mathbb{F}\text{-c.s.a. of degree n} \end{array} \right\}$$
$$[\alpha] \mapsto [\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}]$$

Normal action: $\sigma([a_{ij}]) = [\sigma(a_{ij})]$
Twisted action: $\sigma \cdot_\alpha [a_{ij}] = \alpha_\sigma([\sigma(a_{ij})])$.

$H^1(\mathbb{F}, \mathbf{PSO}_n) \leftrightarrow \{\text{isomorphism classes of } (A, \tau)\}$
$H^1(\mathbb{F}, \mathbf{PSp}_{2n}) \leftrightarrow \{\text{isomorphism classes of } (A, \psi)\}$.

For each c.s.a. A, we choose a cocyle α and identify $A = \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha}$.
Same for (A, τ) and (A, ψ).

Galois Cohomology and Adjoint Groups

Consider $(A, \tau) = (\mathsf{M}_n(\mathbb{F}_{\text{sep}}), \Omega_n^*)^{\Gamma_\alpha}$ with cocycle α. There are actions Γ on $\mathbf{PSO}_n(\mathbb{F}_{\text{sep}}) = \text{Aut}_{\mathbb{F}_{\text{sep}}}(\mathsf{M}_n(\mathbb{F}_{\text{sep}}), \Omega_n^*)$

Normal action: $\sigma(\varphi) = \sigma \circ \varphi \circ \sigma^{-1}$
Twisted action: $\sigma \cdot_\alpha \varphi = \alpha_\sigma \circ \sigma \circ \varphi \circ \sigma^{-1} \circ \alpha_\sigma^{-1}$.

$$\mathbf{PSO}_n(\mathbb{F}_{\text{sep}})^{\Gamma_\alpha} = \text{Aut}_{\mathbb{F}}(A, \tau) = \mathbf{PSO}(A, \tau)(\mathbb{F})$$

Galois Cohomology and Adjoint Groups

$(A, \tau) = (\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \Omega_n^*)^{\Gamma_\alpha}$ with cocycle α.

For $R \in \mathbf{Alg}_{\mathbb{F}}$, take the image $\alpha_\sigma \mapsto \alpha'_\sigma$ via $\mathbf{PSO}_n(\mathbb{F}_{\mathrm{sep}}) \to \mathbf{PSO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$. We act Γ on $\mathbf{PSO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$

Normal action: $\sigma(x) = x'$, where $x' \colon H_{\mathbf{PSO}_n} \xrightarrow{x} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \xrightarrow{1 \otimes \sigma} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$
Twisted action: $\sigma \cdot_\alpha x = \alpha'_\sigma \sigma(x) \alpha'^{-1}_\sigma$.

$$\mathbf{PSO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha} = \mathbf{PSO}(A, \tau)(R)$$

Similarly: (A, ψ) has cocycle β, then

$$\mathbf{PSp}_{2n}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\beta} = \mathbf{PSp}(A, \psi)(R).$$

Galois Cohomology and Other Linear Algebraic Groups

$(A,\tau) = (\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}}), \Omega_n^*)^{\Gamma_\alpha}$ with cocycle α.

If $\mathbf{G}_n$ is any split group isogenous to $\mathbf{PSO}_n$, then $\mathbf{G}_n \twoheadrightarrow \mathbf{PSO}_n$ has a central kernel. We can define an action of Γ on $\mathbf{G}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$

$\mathbf{G}_n(\mathbb{F}_{\mathrm{sep}}) \twoheadrightarrow \mathbf{PSO}_n(\mathbb{F}_{\mathrm{sep}})$ is surjective, so we may choose elements $\mathfrak{a}_\sigma$ lying over α_σ. Then consider the images $\mathfrak{a}_\sigma \mapsto \mathfrak{a}'_\sigma$ via $\mathbf{G}_n(\mathbb{F}_{\mathrm{sep}}) \to \mathbf{G}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$.

Normal action: $\sigma(x) = x'$ where $x' \colon H_{\mathbf{G}_n} \xrightarrow{x} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}} \xrightarrow{1\otimes\sigma} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$.
Twisted action: $\sigma \cdot_\alpha x = \mathfrak{a}'_\sigma \sigma(x) \mathfrak{a}'^{-1}_\sigma$

$$\mathbf{G}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha} = \mathbf{G}(A,\tau)(R)$$

Similarly: (A,ψ) has cocycle β, then

$$\mathbf{Sp}_{2n}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\beta} = \mathbf{Sp}(A,\psi)(R).$$

Split Tensor Product Maps

For $R \in \mathbf{Alg}_{\mathbb{F}}$, we consider

$$\begin{aligned} \mathrm{M}_n(R) \times \mathrm{M}_m(R) &\to \mathrm{M}_{nm}(R) \\ (B_1, B_2) &\to B_1 \otimes B_2 \end{aligned}$$

restricts to group homomorphisms

$$\mathbf{SO}_n \times \mathbf{SO}_m \to \mathbf{SO}_{nm}$$
$$\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \to \mathbf{SO}_{4nm}$$

By a theorem of Borel and Tits, these maps will lift to

$$\mathbf{Spin}_n \times \mathbf{Spin}_m \to \mathbf{Spin}_{nm}$$
$$\mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} \to \mathbf{Spin}_{4nm}$$

By understanding the images of these maps explicitly, when n or m is even we can define maps

$$\mathbf{PSO}_{2n} \times \mathbf{PSO}_{2m} \hookrightarrow \mathbf{HSpin}_{4nm}$$
$$\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \hookrightarrow \mathbf{HSpin}_{4nm}$$

Non-split Tensor Product Maps

There is also a natural tensor product map

$$(A_1, \tau_1) \times (A_2, \tau_2) \to (A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)$$
$$(a_1, a_2) \mapsto a_1 \otimes a_2$$

This also restricts to/induces a map

$$\mathbf{SO}(A_1, \tau_1) \times \mathbf{SO}(A_2, \tau_2) \to \mathbf{SO}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)$$

Similarly

$$\mathbf{Sp}(A_1, \psi_1) \times \mathbf{Sp}(A_2, \psi_2) \to \mathbf{SO}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)$$

Non-split Tensor Product Maps

(A_1, τ_1) with cocycle α_1, (A_2, τ_2) with cocycle α_2, and $(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)$ with cocycle β.

Conveniently, for the previous maps, the map on $R' = R \otimes_{\mathbb{F}} \mathbb{F}_{\mathsf{sep}}$ points is Γ-equivariant.

$$\begin{array}{ccccc} & & \mathbf{HSpin}_{4nm}(R') & \longrightarrow & \mathbf{HSpin}_{4nm}(R') \\ & \nearrow & \downarrow & & \downarrow \\ \mathbf{PSO}_{2n}(R') \times \mathbf{PSO}_{2m}(R') & \longrightarrow & \mathbf{PSO}_{4nm}(R') & \longrightarrow & \mathbf{PSO}_{4nm}(R') \end{array}$$

$$\begin{array}{ccccc} & & \mathfrak{a}' & \longmapsto & \mathfrak{b}' \\ & \nearrow & \downarrow & & \downarrow \\ (\alpha_1', \alpha_2') & \longmapsto & \alpha_1' \otimes \alpha_2' & \longmapsto & \beta' \end{array}$$

Non-split Tensor Product Maps

Then we can restrict to fixed points

$$\mathbf{PSO}_{2n}(R')^{\Gamma_{\alpha_1}} \times \mathbf{PSO}_{2m}(R')^{\Gamma_{\alpha_2}} \hookrightarrow \mathbf{HSpin}_{4nm}(R')^{\Gamma_{\beta}}$$

which gives

$$\mathbf{PSO}(A_1, \tau_1) \times \mathbf{PSO}(A_2, \tau_2) \hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)$$

Similarly, we can construct

$$\mathbf{PSp}(A_1, \psi_1) \times \mathbf{PSp}(A_2, \psi_2) \hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)$$

Applications

Using the split map $\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \hookrightarrow \mathbf{HSpin}_{4nm}$ with results of Bermudez-Ruozzi and Merkurjev, it is possible to compute

$$\mathrm{Inv}^3(\mathbf{HSpin}_{4d}, \mathbb{Q}/\mathbb{Z}(2))$$

the group of degree three cohomological invariants. These are functors $H^1(-, \mathbf{HSpin}_{4d}) \to H^3(-, \mathbb{Q}/\mathbb{Z}(2))$.

Bermudez-Ruozzi: $\mathrm{Inv}^3(\mathbf{HSpin}_{4d}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$
Merkurjev: $\mathrm{Inv}^3(\mathbf{PSp}_{2n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$ and also $\mathrm{Inv}^3(\mathbf{PSp}(A, \psi), \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$

The hope is that
$\mathbf{PSp}(A_1, \psi_1) \times \mathbf{PSp}(A_2, \psi_2) \hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)$ will be fruitful in computing

$$\mathrm{Inv}^3(\mathbf{HSpin}(A, \tau), \mathbb{Q}/\mathbb{Z}(2)).$$

Thank You