

Extending Tensor Product Maps to Non-split Linear Algebraic Groups

Cameron Ruether

University of Ottawa

Algebra and Geometry of Homogeneous Spaces

June 4, 2021

Linear Algebraic Groups

Let \mathbb{F} be a field, $\text{char}(\mathbb{F}) \neq 2$. A *linear algebraic group* is a covariant functor

$$\mathbf{G}: \mathbf{Alg}_{\mathbb{F}} \rightarrow \mathbf{Grp}$$
$$R \mapsto \text{Hom}_{\mathbb{F}}(H, R)$$

where H is a finitely presented \mathbb{F} -Hopf algebra.

A homomorphism of linear algebraic groups is a natural transformation

$$\varphi: \mathbf{G}_1 \rightarrow \mathbf{G}_2$$
$$\varphi(R): \mathbf{G}_1(R) \rightarrow \mathbf{G}_2(R).$$

Examples

$$\Omega_n = \begin{bmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{bmatrix}, \Psi_{2n} = \begin{bmatrix} & \Omega_n \\ -\Omega_n & \end{bmatrix}$$

Split

$$\mathbf{SL}_n(R) = \{B \in M_n(R) \mid \det(B) = 1\}$$

$$\mathbf{SO}_n(R) = \{B \in M_n(R) \mid B\Omega_n B^T \Omega_n = I, \det(B) = 1\}$$

$$\mathbf{Sp}_{2n}(R) = \{B \in M_{2n}(R) \mid B\Psi_{2n} B^T \Psi_{2n} = I\}$$

Non-split

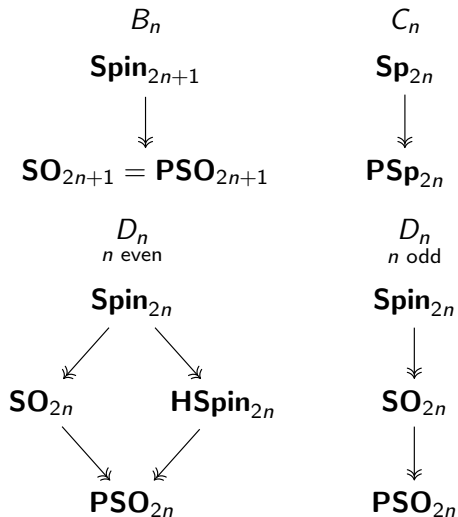
$$\mathbf{SL}(A)(R) = \{a \in A \otimes_{\mathbb{F}} R \mid \text{Nrd}(a) = 1\}$$

$$\mathbf{SO}(A, \tau)(R) = \{a \in A \otimes_{\mathbb{F}} R \mid a \cdot \tau(a) = 1, \text{Nrd}(a) = 1\}$$

$$\mathbf{Sp}(A, \psi)(R) = \{a \in A \otimes_{\mathbb{F}} R \mid a \cdot \psi(a) = 1\}$$

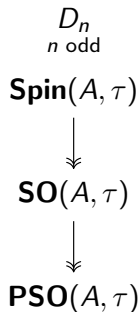
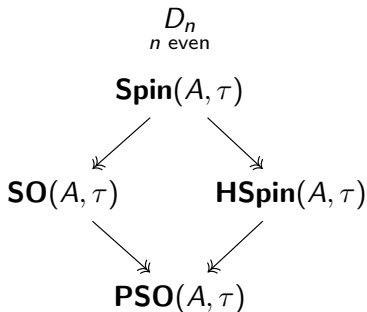
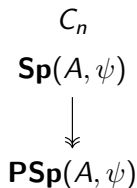
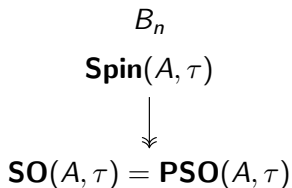
A is a central simple \mathbb{F} -algebra, τ an orthogonal involution, ψ a symplectic involution.

Split Types B, C, and D



n even: $Z(\mathbf{Spin}_{2n}) \cong \mu_2 \times \mu_2$. n odd: $Z(\mathbf{Spin}_{2n}) \cong \mu_4$.

Non-split Types B, C, and D



Galois Cohomology and Central Simple Algebra

Let $\Gamma = \text{Gal}(\mathbb{F}_{\text{sep}}/\mathbb{F})$.

We use the Galois cohomology sets $H^1(\mathbb{F}, \mathbf{G}) := H^1(\Gamma, \mathbf{G}(\mathbb{F}_{\text{sep}}))$.

If A is an \mathbb{F} -c.s.a of degree n , then $A \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}} \cong M_n(\mathbb{F}_{\text{sep}})$.

$$H^1(\mathbb{F}, \mathbf{PGL}_n) \leftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \mathbb{F}\text{-c.s.a. of degree } n \end{array} \right\}$$
$$[\alpha] \mapsto [M_n(\mathbb{F}_{\text{sep}})^{\Gamma_\alpha}]$$

Normal action: $\sigma([a_{ij}]) = [\sigma(a_{ij})]$

Twisted action: $\sigma \cdot_\alpha [a_{ij}] = \alpha_\sigma([\sigma(a_{ij})])$.

$H^1(\mathbb{F}, \mathbf{PSO}_n) \leftrightarrow \{\text{isomorphism classes of } (A, \tau)\}$

$H^1(\mathbb{F}, \mathbf{PSp}_{2n}) \leftrightarrow \{\text{isomorphism classes of } (A, \psi)\}$.

For each c.s.a. A , we choose a cocycle α and identify $A = M_n(\mathbb{F}_{\text{sep}})^{\Gamma_\alpha}$.

Same for (A, τ) and (A, ψ) .

Galois Cohomology and Adjoint Groups

Consider $(A, \tau) = (M_n(\mathbb{F}_{\text{sep}}), \Omega_n^*)^{\Gamma_\alpha}$ with cocycle α . There are actions Γ on $\mathbf{PSO}_n(\mathbb{F}_{\text{sep}}) = \text{Aut}_{\mathbb{F}_{\text{sep}}}(M_n(\mathbb{F}_{\text{sep}}), \Omega_n^*)$

Normal action: $\sigma(\varphi) = \sigma \circ \varphi \circ \sigma^{-1}$

Twisted action: $\sigma \cdot_\alpha \varphi = \alpha_\sigma \circ \sigma \circ \varphi \circ \sigma^{-1} \circ \alpha_\sigma^{-1}$.

$$\mathbf{PSO}_n(\mathbb{F}_{\text{sep}})^{\Gamma_\alpha} = \text{Aut}_{\mathbb{F}}(A, \tau) = \mathbf{PSO}(A, \tau)(\mathbb{F})$$

Galois Cohomology and Adjoint Groups

$(A, \tau) = (M_n(\mathbb{F}_{\text{sep}}), \Omega_n^*)^{\Gamma_\alpha}$ with cocycle α .

For $R \in \mathbf{Alg}_{\mathbb{F}}$, take the image $\alpha_\sigma \mapsto \alpha'_\sigma$ via $\mathbf{PSO}_n(\mathbb{F}_{\text{sep}}) \rightarrow \mathbf{PSO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})$. We act Γ on $\mathbf{PSO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})$

Normal action: $\sigma(x) = x'$, where $x': H_{\mathbf{PSO}_n} \xrightarrow{x} R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}} \xrightarrow{1 \otimes \sigma} R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}$
Twisted action: $\sigma \cdot_\alpha x = \alpha'_\sigma \sigma(x) \alpha'^{-1}_\sigma$.

$$\mathbf{PSO}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^{\Gamma_\alpha} = \mathbf{PSO}(A, \tau)(R)$$

Similarly: (A, ψ) has cocycle β , then

$$\mathbf{PSp}_{2n}(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^{\Gamma_\beta} = \mathbf{PSp}(A, \psi)(R).$$

Galois Cohomology and Other Linear Algebraic Groups

$(A, \tau) = (M_n(\mathbb{F}_{\text{sep}}), \Omega_n^*)^{\Gamma_\alpha}$ with cocycle α .

If \mathbf{G}_n is any split group isogenous to \mathbf{PSO}_n , then $\mathbf{G}_n \rightarrow \mathbf{PSO}_n$ has a central kernel. We can define an action of Γ on $\mathbf{G}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})$

$\mathbf{G}_n(\mathbb{F}_{\text{sep}}) \rightarrow \mathbf{PSO}_n(\mathbb{F}_{\text{sep}})$ is surjective, so we may choose elements \mathfrak{a}_σ lying over α_σ . Then consider the images $\mathfrak{a}_\sigma \mapsto \mathfrak{a}'_\sigma$ via

$\mathbf{G}_n(\mathbb{F}_{\text{sep}}) \rightarrow \mathbf{G}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})$.

Normal action: $\sigma(x) = x'$ where $x': H_{\mathbf{G}_n} \xrightarrow{x} R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}} \xrightarrow{1 \otimes \sigma} R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}$.

Twisted action: $\sigma \cdot_\alpha x = \mathfrak{a}'_\sigma \sigma(x) \mathfrak{a}'_\sigma^{-1}$

$$\mathbf{G}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^{\Gamma_\alpha} = \mathbf{G}(A, \tau)(R)$$

Similarly: (A, ψ) has cocycle β , then

$$\mathbf{Sp}_{2n}(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^{\Gamma_\beta} = \mathbf{Sp}(A, \psi)(R).$$

Split Tensor Product Maps

For $R \in \mathbf{Alg}_{\mathbb{F}}$, we consider

$$\begin{aligned} M_n(R) \times M_m(R) &\rightarrow M_{nm}(R) \\ (B_1, B_2) &\rightarrow B_1 \otimes B_2 \end{aligned}$$

restricts to group homomorphisms

$$\begin{aligned} \mathbf{SO}_n \times \mathbf{SO}_m &\rightarrow \mathbf{SO}_{nm} \\ \mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} &\rightarrow \mathbf{SO}_{4nm} \end{aligned}$$

By a theorem of Borel and Tits, these maps will lift to

$$\begin{aligned} \mathbf{Spin}_n \times \mathbf{Spin}_m &\rightarrow \mathbf{Spin}_{nm} \\ \mathbf{Sp}_{2n} \times \mathbf{Sp}_{2m} &\rightarrow \mathbf{Spin}_{4nm} \end{aligned}$$

By understanding the images of these maps explicitly, when n or m is even we can define maps

$$\begin{aligned} \mathbf{PSO}_{2n} \times \mathbf{PSO}_{2m} &\hookrightarrow \mathbf{HSpin}_{4nm} \\ \mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} &\hookrightarrow \mathbf{HSpin}_{4nm} \end{aligned}$$

Non-split Tensor Product Maps

There is also a natural tensor product map

$$\begin{aligned}(A_1, \tau_1) \times (A_2, \tau_2) &\rightarrow (A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2) \\ (a_1, a_2) &\mapsto a_1 \otimes a_2\end{aligned}$$

This also restricts to/induces a map

$$\mathbf{SO}(A_1, \tau_1) \times \mathbf{SO}(A_2, \tau_2) \rightarrow \mathbf{SO}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)$$

Similarly

$$\mathbf{Sp}(A_1, \psi_1) \times \mathbf{Sp}(A_2, \psi_2) \rightarrow \mathbf{SO}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)$$

Non-split Tensor Product Maps

(A_1, τ_1) with cocycle α_1 , (A_2, τ_2) with cocycle α_2 , and $(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)$ with cocycle β .

Conveniently, for the previous maps, the map on $R' = R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}$ points is Γ -equivariant.

$$\begin{array}{ccccc}
 & & \mathbf{HSpin}_{4nm}(R') & \longrightarrow & \mathbf{HSpin}_{4nm}(R') \\
 & \nearrow & \downarrow & & \downarrow \\
 \mathbf{PSO}_{2n}(R') \times \mathbf{PSO}_{2m}(R') & \longrightarrow & \mathbf{PSO}_{4nm}(R') & \longrightarrow & \mathbf{PSO}_{4nm}(R')
 \end{array}$$

$$\begin{array}{ccccc}
 & & \mathbf{a}' & \longrightarrow & \mathbf{b}' \\
 & \nearrow & \downarrow & & \downarrow \\
 (\alpha'_1, \alpha'_2) & \longrightarrow & \alpha'_1 \otimes \alpha'_2 & \longrightarrow & \beta'
 \end{array}$$

Non-split Tensor Product Maps

Then we can restrict to fixed points

$$\mathbf{PSO}_{2n}(R')^{\Gamma_{\alpha_1}} \times \mathbf{PSO}_{2m}(R')^{\Gamma_{\alpha_2}} \hookrightarrow \mathbf{HSpin}_{4nm}(R')^{\Gamma_{\beta}}$$

which gives

$$\mathbf{PSO}(A_1, \tau_1) \times \mathbf{PSO}(A_2, \tau_2) \hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \tau_1 \otimes \tau_2)$$

Similarly, we can construct

$$\mathbf{PSp}(A_1, \psi_1) \times \mathbf{PSp}(A_2, \psi_2) \hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)$$

Applications

Using the split map $\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \hookrightarrow \mathbf{HSpin}_{4nm}$ with results of Bermudez-Ruozzi and Merkurjev, it is possible to compute

$$\mathrm{Inv}^3(\mathbf{HSpin}_{4d}, \mathbb{Q}/\mathbb{Z}(2))$$

the group of degree three cohomological invariants. These are functors $H^1(-, \mathbf{HSpin}_{4d}) \rightarrow H^3(-, \mathbb{Q}/\mathbb{Z}(2))$.

Bermudez-Ruozzi: $\mathrm{Inv}^3(\mathbf{HSpin}_{4d}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$

Merkurjev: $\mathrm{Inv}^3(\mathbf{PSp}_{2n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$ and also $\mathrm{Inv}^3(\mathbf{PSp}(A, \psi), \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$

The hope is that

$\mathbf{PSp}(A_1, \psi_1) \times \mathbf{PSp}(A_2, \psi_2) \hookrightarrow \mathbf{HSpin}(A_1 \otimes_{\mathbb{F}} A_2, \psi_1 \otimes \psi_2)$ will be fruitful in computing

$$\mathrm{Inv}^3(\mathbf{HSpin}(A, \tau), \mathbb{Q}/\mathbb{Z}(2)).$$

Thank You

