Twisting Linear Algebraic Groups and Hopf Algebras

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Central Simple Algebras

Let $\mathbb{F}$ be a field, $\text{char}(\mathbb{F}) \neq 2$. Let $\mathbb{F}_{\text{sep}}$ be a separable closure, and $\Gamma = \text{Gal}(\mathbb{F}_{\text{sep}}/\mathbb{F})$.
Let A be a central simple algebra over F, then by Wedderburn's theorem

$$A \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}} \cong \text{M}_n(\mathbb{F}_{\text{sep}}).$$

Example

$\mathbb{H} = \text{Span}_{\mathbb{R}}(\{1, I, J, K\})$ *with multiplication given by* $I^2 = J^2 = K^2 = -1$, $IJ = K = -JI$.
Then $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \text{M}_2(\mathbb{C})$ *via*

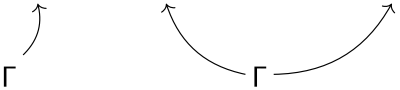
$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$J \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad K \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Central Simple Algebras

There are two Γ–actions

$$A \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}} \cong \mathrm{M}_n(\mathbb{F}_{\text{sep}}) \cong \mathrm{M}_n(\mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}}$$



The difference between these actions is measured by a 1-cocycle $\alpha \colon \Gamma \to \mathrm{Aut}_{\mathbb{F}_{\text{sep}}}(\mathrm{M}_n(\mathbb{F}_{\text{sep}})) \cong \mathbf{PGL}_n(\mathbb{F}_{\text{sep}})$, $\alpha_{\sigma\tau} = \alpha_\sigma \sigma(\alpha_\tau)$. Define

$$\sigma \cdot_\alpha x := \alpha_\sigma(\sigma(x))$$

Then, $\mathrm{M}_n(\mathbb{F}_{\text{sep}})^{\Gamma_\alpha} \cong A$.

Example

$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{M}_2(\mathbb{C})$. *The cocycle* $\alpha \colon \Gamma \to \mathbf{PGL}_2(\mathbb{C})$ *is given by* $\sigma \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. *Then* $\sigma \cdot_\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \sigma(d) & -\sigma(c) \\ -\sigma(b) & \sigma(a) \end{bmatrix}$.

Central Simple Algebras

$$\left\{\begin{array}{c}\text{Iso. classes of}\\ \text{degree n } \mathbb{F}\text{–c.s.a.}\end{array}\right\} \leftrightarrow H^1(\mathbb{F}, \mathbf{PGL}_n)$$

$$\left\{\begin{array}{c}\text{Iso. classes of}\\ \text{degree 2n } \mathbb{F}\text{–c.s.a.}\\ \text{w/ symplectic involution}\end{array}\right\} \leftrightarrow H^1(\mathbb{F}, \mathbf{PSp}_{2n})$$

$$\left\{\begin{array}{c}\text{Iso. classes of}\\ \text{degree n } \mathbb{F}\text{–c.s.a.}\\ \text{w/ orthogonal involution}\\ \text{of trivial discriminant}\end{array}\right\} \leftrightarrow H^1(\mathbb{F}, \mathbf{PSO}_n)$$

Difference in Γ–action

Twisted Forms ⇄ Cocycles

Fixed points of twisted Γ–actions

Linear Algebraic Groups

A linear algebraic group over $\mathbb{F}$ is a covariant functor

$$\begin{aligned} \mathbf{G}\colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp} \\ R &\mapsto \mathrm{Hom}_{\mathbb{F}}(H, R) \end{aligned}$$

represented by an $\mathbb{F}$-Hopf algebra H.

$$\begin{aligned} \mathbf{GL}(A)\colon R &\mapsto (A \otimes_{\mathbb{F}} R)^{\times} \\ \mathbf{SL}(A)\colon R &\mapsto \{x \in A \otimes_{\mathbb{F}} R \mid \mathrm{Nrd}(x) = 1\} \\ \mathbf{PGL}(A)\colon R &\mapsto \mathrm{Aut}_R(A \otimes_{\mathbb{F}} R) \\ \mathbf{SO}(A,\tau)\colon R &\mapsto \{x \in A \otimes_{\mathbb{F}} R \mid \mathrm{Nrd}(x) = 1, x \cdot (\tau \otimes 1)(x) = 1\} \\ \mathbf{PSO}(A,\tau)\colon R &\mapsto \{\varphi \in \mathrm{Aut}_R(A \otimes_{\mathbb{F}} R, \tau \otimes 1) \mid \gamma(\varphi) = \mathrm{id}\} \end{aligned}$$

$\gamma\colon \mathrm{Aut}_R(A \otimes_{\mathbb{F}} R, \tau \otimes 1) \to \mathrm{Aut}_R(C(A,\tau) \otimes R) \to \mathrm{Aut}_R(Z(C(A,\tau) \otimes R))$.

Split groups: $\mathbf{SL}(\mathrm{M}_n(\mathbb{F})) = \mathbf{SL}_n$, $\mathbf{SO}(\mathrm{M}_n(\mathbb{F}), \tau_0) = \mathbf{SO}_n$, etc.

Type D

$$\begin{array}{ccc} & \mathbf{Spin}(A,\tau) & \\ \swarrow & & \searrow \\ \mathbf{SO}(A,\tau) & & \mathbf{HSpin}(A,\tau) \\ \searrow & & \swarrow \\ & \mathbf{PSO}(A,\tau) & \end{array}$$

$$\mathbf{Spin}(A,\tau)\colon R \mapsto \left\{ x \in C(A,\tau) \otimes_{\mathbb{F}} R \;\middle|\; \begin{array}{l} x \cdot (\tau \otimes 1)(x) = 1, \\ x * (b(A) \otimes_{\mathbb{F}} R) \bullet x^{-1} = b(A) \otimes_{\mathbb{F}} R \end{array} \right\}$$

When $4 \mid \deg(A)$, $C(A,\tau) = C^+(A,\tau) \times C^-(A,\tau)$. So we have

$$\mathbf{Spin}(A,\tau) \to \mathbf{GL}(C(A,\tau)) \to \mathbf{GL}(C^+(A,\tau)).$$

The scheme-theoretic image of this map is $\mathbf{HSpin}(A,\tau)$. In terms of Hopf algebras, $H_{\mathbf{HSpin}(A,\tau)}$ is the image of

$$H_{\mathbf{GL}(C^+(A,\tau))} \to H_{\mathbf{GL}(C(A,\tau))} \to H_{\mathbf{Spin}(A,\tau)}.$$

Twisted Forms

Let $\mathbf{G}(A)$ be any of the previous groups, and $\mathbf{G}_n$ the split version.

$$\mathbf{G}(A)(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathsf{sep}}) \cong \mathbf{G}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathsf{sep}}).$$

Standard Galois actions: For any group $\mathbf{G}$, let $x \in \mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathsf{sep}})$, $\sigma \in \Gamma$,

$$\sigma(x)\colon H_{\mathbf{G}} \xrightarrow{x} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathsf{sep}} \xrightarrow{1\otimes\sigma} R \otimes_{\mathbb{F}} \mathbb{F}_{\mathsf{sep}}$$

and then $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathsf{sep}})^{\Gamma} = \mathbf{G}(R)$.

So, we expect $\mathbf{G}(A)$ to correspond to some cocycle in $H^1(\mathbb{F}, \mathrm{Aut}(\mathbf{G}_n))$.

Twisting Groups

Let A be an $\mathbb{F}$–c.s.a. with corresponding cocycle $\alpha\colon \Gamma \to \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$. For $x \in \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$, define

$$\sigma \cdot_\alpha x = \alpha_\sigma \sigma(x) \alpha_\sigma^{-1}.$$

Then $\mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha} = \mathbf{PGL}(A)(\mathbb{F})$.

Further, let $\alpha_\sigma \mapsto \alpha'_\sigma$ under $\mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}}) \to \mathbf{PGL}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$, define

$$\sigma \cdot_\alpha x = \alpha'_\sigma \sigma(x) (\alpha'_\sigma)^{-1}$$

Then $\mathbf{PGL}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha} = \mathbf{PGL}(A)(R)$.

Twisting Groups

Choose elements $a_\sigma \in \mathbf{SL}_n(\mathbb{F}_{\text{sep}})$ such that $a_\sigma \mapsto \alpha_\sigma$ via $\mathbf{SL}_n(\mathbb{F}_{\text{sep}}) \to \mathbf{PGL}_n(\mathbb{F}_{\text{sep}})$.
Similarly denote $a_\sigma \mapsto a'_\sigma$ under $\mathbf{SL}_n(\mathbb{F}_{\text{sep}}) \to \mathbf{SL}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})$. Twist by conjugation,

$$\sigma \cdot_\alpha x := a'_\sigma \sigma(x)(a'_\sigma)^{-1}.$$

Then $\mathbf{SL}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\text{sep}})^{\Gamma_\alpha} = \mathbf{SL}(A)(R)$.

Theorem

The same thing works for all simple classical groups.

The proof is computational for all groups except **HSpin**.

Twisting Hopf Algebras

Setup: $\mathbf{G}_n$ split group, $\mathbf{G}(A)$ twisted form. Cocycle $\alpha\colon \Gamma \to \mathrm{Inn}(\mathbf{G}_n(\mathbb{F}_{\mathrm{sep}}))$, $\alpha_\sigma(x) = a_\sigma x a_\sigma^{-1}$. Twist by this cocycle, then $\mathbf{G}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\Gamma_\alpha} = \mathbf{G}(A)(R)$.

Let H be the Hopf algebra of $\mathbf{G}_n$, set $H_{\mathrm{sep}} = H \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}$.

$$a_\sigma \in \mathrm{Hom}_{\mathbb{F}}(H, \mathbb{F}_{\mathrm{sep}}) \cong \mathrm{Hom}_{\mathbb{F}_{\mathrm{sep}}}(H_{\mathrm{sep}}, \mathbb{F}_{\mathrm{sep}}).$$

Define $\texttt{@}_\sigma \in \mathrm{Aut}_{\mathbb{F}_{\mathrm{sep}}}(H_{\mathrm{sep}})$ by

$$\texttt{@}_\sigma\colon H_{\mathrm{sep}} \xrightarrow{c^2} H_{\mathrm{sep}} \otimes H_{\mathrm{sep}} \otimes H_{\mathrm{sep}} \xrightarrow{a_\sigma \otimes \mathrm{id} \otimes a_\sigma^{-1}} \mathbb{F}_{\mathrm{sep}} \otimes H_{\mathrm{sep}} \otimes \mathbb{F}_{\mathrm{sep}} \xrightarrow{m} H_{\mathrm{sep}}.$$

Then for $x \in \mathbf{G}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$,

$$x \circ \texttt{@}_\sigma = a'_\sigma x (a'_\sigma)^{-1}$$

Twisting Hopf Algebras

Define

$$\begin{aligned}\mathfrak{A}\colon \Gamma &\to \mathrm{Aut}_{\mathbb{F}_{\mathrm{sep}}}(H_{\mathrm{sep}})\\ \sigma &\mapsto \sigma \circ \mathbb{C}_{\sigma^{-1}} \circ \sigma^{-1},\end{aligned}$$

Then twist as usual for $x \in H_{\mathrm{sep}}$,

$$\sigma \cdot_{\alpha} x := \mathfrak{A}_{\sigma}\sigma(x)$$

and we obtain $(H_{\mathrm{sep}})^{\Gamma_{\alpha}} = H(A)$, the $\mathbb{F}$–Hopf algebra of $\mathbf{G}(A)$.

Twisting **HSpin**

$$
\begin{array}{ccccc}
\mathbf{Spin}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) & \longrightarrow & \mathbf{GL}(C_n)(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) & \longrightarrow & \mathbf{GL}(C_n^+)(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{Spin}(A,\tau)(R) & \longrightarrow & \mathbf{GL}(C(A,\tau))(R) & \longrightarrow & \mathbf{GL}(C^+(A,\tau))(R)
\end{array}
$$

$$
\begin{array}{ccccc}
H_{C_n^+,\mathrm{sep}} & \longrightarrow & H_{C_n,\mathrm{sep}} & \longrightarrow & H_{\mathbf{Spin},\mathrm{sep}} \\
\uparrow & & \uparrow & & \uparrow \\
H_{C^+(A,\tau)} & \longrightarrow & H_{C(A,\tau)} & \longrightarrow & H_{\mathbf{Spin}(A,\tau)}
\end{array}
$$

$$\varphi : H_{C_n^+,\mathrm{sep}} \dashrightarrow H_{\mathbf{Spin},\mathrm{sep}}, \qquad \phi : H_{C^+(A,\tau)} \dashrightarrow H_{\mathbf{Spin},\mathrm{sep}}$$

Then $\mathrm{Img}(\varphi)^{\Gamma_\alpha} = \mathrm{Img}(\phi) \Leftrightarrow (H_{\mathbf{HSpin}_n,\mathrm{sep}})^{\Gamma_\alpha} = H_{\mathbf{HSpin}(A,\tau)}$.

Thank You