The Norm Functor over Schemes

Cameron Ruether
Joint with Philippe Gille and Erhard Neher

Memorial University of Newfoundland

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Corestriction

- Riehm: "The Corestriction of Algebraic Structures" (1970)
  - $\mathbb{K}/\mathbb{F}$ a finite separable field extension.
  - For a $\mathbb{K}$–object X, functorially assigns an $\mathbb{F}$-object $\mathsf{Cor}(X)$.
  - For central simple algebras, we get a diagram

$$\begin{array}{ccc} \mathsf{Br}(\mathbb{K}) & \xrightarrow{\sim} & H^2(\mathsf{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{K}), \mathbb{F}_{\mathrm{sep}}^{\times}) \\ \downarrow{\scriptstyle [\mathsf{Cor}]} & & \downarrow{\scriptstyle \mathrm{cor}} \\ \mathsf{Br}(\mathbb{F}) & \xrightarrow{\sim} & H^2(\mathsf{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F}), \mathbb{F}_{\mathrm{sep}}^{\times}) \end{array}$$

Norm for Étale Extensions

- Knus & Ojanguren: "A Norm for Modules and Algebras" (1975)
  - S/R a finite étale ring extension.
  - Define a functor $N_{S/R} \colon S\text{–}\mathrm{Mod} \to R\text{–}\mathrm{Mod}$.
  - If $S = R \times \ldots \times R$, an S–module is $M_1 \times \ldots \times M_d$, and

$$N_{S/R}(M_1 \times \ldots \times M_d) = M_1 \otimes_R \ldots \otimes_R M_d$$

  - Since S/R is finite étale, the general case is obtained from above by faithfully flat descent.
  - $N_{S/R}$ also preserves Azumaya algebras and induces a homomorphism $\mathrm{Br}(S) \to \mathrm{Br}(R)$.

Ferrand's Norm Functor

- Ferrand: "Un Foncteur Norme" (1998)
  - S/R a finite locally free extension of rings.
  - Defines a functor $N_{S/R}\colon S\text{–}\mathrm{Mod} \to R\text{–}\mathrm{Mod}$ which generalizes the previous two constructions.
  - The construction does not use descent, instead based on *polynomial laws*.

For two R–modules M_1, M_2, a *polynomial law* between them is a natural transformation

$$\nu\colon \mathbf{W}(M_1) \to \mathbf{W}(M_2)$$

where $\mathbf{W}(M)$ is the functor

$$\begin{aligned} \mathbf{W}(M)\colon \mathfrak{Alg}_R &\to \mathfrak{Ab} \\ R' &\mapsto M \otimes_R R'. \end{aligned}$$

Ferrand's Norm Functor

Since S/R is finite locally free, we have $\det\colon S \otimes_R R' \to R'$ and a polynomial law

$$\begin{aligned} \mathrm{norm}\colon \mathbf{W}({}_R S) &\to \mathbf{W}(R) \\ s \otimes r' &\mapsto \det(s \otimes r'). \end{aligned}$$

- Properties of $N_{S/R}$:
  - For $M \in S\text{–}\mathrm{Mod}$, there is a polynomial law $\nu_M\colon \mathbf{W}({}_R M) \to \mathbf{W}(N_{S/R}(M))$ satisfying

    $$\nu_M((s \otimes r')m') = \mathrm{norm}(s \otimes r')\nu_M(m').$$

  - ν_M is universal. If $\nu'\colon \mathbf{W}({}_R M) \to \mathbf{W}(M')$ is also "norm semi-linear" then $\exists!$ R–module map $\varphi\colon N_{S/R}(M) \to M'$ such that $\nu' = \varphi \circ \nu_M$.

Ferrand's Norm Functor

- Properties of $N_{S/R}$:
  - $N_{S/R}(S) = R$ and $\nu_S = \mathrm{norm}$.
  - If $S = R \times \ldots \times R$, then $N_{S/R}(M_1 \times \ldots \times M_d) = M_1 \otimes_R \ldots \otimes_R M_d$ and
  $$\nu_M(m_1, \ldots, m_d) = m_1 \otimes \ldots \otimes m_d.$$
  - For $R' \in \mathfrak{Alg}_R$ and $M \in S\text{–}\mathrm{Mod}$,
  $$N_{(S \otimes_R R')/R'}(M \otimes_R R') \cong N_{S/R}(M) \otimes_R R'.$$

Setting over a Scheme

Let S be an arbitrary base scheme.

- Work with sheaves on $\mathfrak{Sch}_S$ for the fppf topology.
- Modules, algebras, etc., are over

$$\begin{aligned} \mathcal{O}\colon \mathfrak{Sch}_S &\to \mathfrak{Ab} \\ T &\mapsto \mathcal{O}_T(T). \end{aligned}$$

In this context: An $\mathcal{O}$–module $\mathcal{M}\colon \mathfrak{Sch}_S \to \mathfrak{Ab}$ is quasi-coherent if and only if for every morphism $V \to U \in \mathfrak{Sch}_S$ where U, V are affine schemes,

$$\mathcal{M}(V) \cong \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(V).$$

Globalizing the Norm Functor

Let $f\colon T \to S$ be a finite locally free morphism of schemes.
$\Rightarrow \forall S' \in \mathfrak{Sch}_S$,

$$\mathcal{O}(S') \to \mathcal{O}(T \times_S S')$$

is a finite locally free ring extension. This means we have a norm natural transformation

$$\mathrm{norm}\colon f_*(\mathcal{O}|_T) \to \mathcal{O}.$$

We want to define a functor $N_{T/S}\colon \mathfrak{QCoh}_T \to \mathfrak{QCoh}_S$ which generalizes Ferrand's norm.

Lemma (Stacks Project Tags 021V, 03DM)

There is an equivalence of categories between sheaves on $\mathfrak{Sch}_S$ and sheaves on $\mathfrak{Aff}_S$ which preserves quasi-coherence.

Globalizing the Norm Functor

For $\mathcal{M} \in \mathfrak{QCoh}_T$, define

$$\begin{aligned} N_{T/S}(\mathcal{M})\colon \mathfrak{Aff}_S &\to \mathfrak{Ab} \\ U &\mapsto N_{\mathcal{O}(T\times_S U)/\mathcal{O}(U)}(\mathcal{M}(T\times_S U)). \end{aligned}$$

which will be a quasi-coherent sheaf due to the properties of Ferrand's norm.

Furthermore, the universal polynomial laws

$$\nu_{\mathcal{M}(T\times_S U)}\colon \mathbf{W}({}_{\mathcal{O}(U)}\mathcal{M}(T\times_S U)) \to \mathbf{W}\big(N_{\mathcal{O}(T\times_S U)/\mathcal{O}(U)}(\mathcal{M}(T\times_S U))\big)$$

assemble into a natural transformation $\nu_{\mathcal{M}}\colon f_*(\mathcal{M}) \to N_{T/S}(\mathcal{M})$ satisfying

$$\nu_{\mathcal{M}}(tm) = \mathrm{norm}(t)\nu_{\mathcal{M}}(m)$$

for all $t \in f_*(\mathcal{O}|_T)$ and $m \in f_*(\mathcal{M})$.

Properties of the Norm Functor

We get a sheaf $N_{T/S}(\mathcal{M})\colon \mathfrak{Sch}_S \to \mathfrak{Ab}$ and a natural transformation $\nu_M\colon f_*(\mathcal{M}) \to N_{T/S}(\mathcal{M})$.

- $N_{T/S}(\mathcal{M})$ is quasi-coherent by construction.
- ν_M is universal. If $\mathcal{M}' \in \mathfrak{QCoh}_S$ and $\nu\colon f_*(\mathcal{M}) \to \mathcal{M}'$ with $\nu(tm) = \mathrm{norm}(t)\nu(m)$, then $\exists!$ $\mathcal{O}$–module morphism $\varphi\colon N_{T/S}(\mathcal{M}) \to \mathcal{M}'$ such that $\nu = \varphi \circ \nu_{\mathcal{M}}$.
- If $T = S \sqcup \ldots \sqcup S$ then

$$N_{T/S}(\mathcal{M}_1, \ldots, \mathcal{M}_d) = \mathcal{M}_1 \otimes_{\mathcal{O}} \ldots \otimes_{\mathcal{O}} \mathcal{M}_d.$$

Furthermore, if $T \to S$ is a finite étale cover of degree d, then

- $\mathcal{M}$ finite locally free of rank $r \Rightarrow N_{T/S}(\mathcal{M})$ finite locally free of rank r^d.
- $\mathcal{A}$ is an Azumaya $\mathcal{O}|_T$–algebra of degree $r \Rightarrow N_{T/S}(\mathcal{A})$ is an Azumaya $\mathcal{O}$–algebra of degree r^d.

The Norm Stack Morphism

We get a functor $N_{T/S}\colon \mathfrak{QCoh}_T \to \mathfrak{QCoh}_S$.

Analogously, for any $S' \in \mathfrak{Sch}_S$, we get a functor $N_{(T\times_S S')/S'}\colon \mathfrak{QCoh}_{T\times_S S'} \to \mathfrak{QCoh}_S$. These fit together into a morphism of stacks.

Let $p\colon \mathfrak{QCoh}_{\mathrm{flf}} \to \mathfrak{Sch}_S$ be the stack with objects $(T' \to S', \mathcal{M}')$ where

- $T' \to S'$ is a finite locally free morphism in $\mathfrak{Sch}_S$,
- $\mathcal{M}'$ is a quasi-coherent $\mathcal{O}|_{T'}$–module,
- $p(T' \to S', \mathcal{M}') = S'$.

Theorem

We have a norm stack morphism

$$\begin{aligned} N\colon \mathfrak{QCoh}_{\mathrm{flf}} &\to \mathfrak{QCoh} \\ (T' \to S', \mathcal{M}') &\mapsto (S', N_{T'/S'}(\mathcal{M}')). \end{aligned}$$

Cohomology of the Norm

Lemma (Giraud "Cohomologie non Abélienne")

If $\varphi\colon \mathfrak{F} \to \mathfrak{G}$ *is a morphism of gerbes over* $\mathfrak{Sch}_S$ *and* $x \in \mathfrak{F}(S)$*, then the associated group homomorphism* $\varphi^*\colon \mathbf{Aut}(x) \to \mathbf{Aut}(\varphi(x))$ *induces the following map on cohomology*

$$H^1(S, \mathbf{Aut}(x)) \to H^1(S, \mathbf{Aut}(\varphi(x)))$$
$$[x'] \mapsto [\varphi(x')]$$

since $H^1(S, \mathbf{Aut}(x)) = \{\textit{Isomorphism classes in } \mathfrak{F}(S)\}$ *and* $H^1(S, \mathbf{Aut}(\varphi(x))) = \{\textit{Isomorphism classes in } \mathfrak{G}(S)\}$.

Cohomology of the Norm

Consider the groupoid $\mathfrak{Mod}_r^{d\text{-ét}}$ of

- pairs $(T \to S, \mathcal{M})$ where $T \to S$ is finite étale of degree d and $\mathcal{M}$ is a locally free $\mathcal{O}|_T$–module of rank r,
- morphisms $(g, \varphi)\colon (T' \to S, \mathcal{M}') \to (T \to S, \mathcal{M})$ where $g\colon T' \to T$ and $\varphi\colon \mathcal{M}' \xrightarrow{\sim} g^*(\mathcal{M})$.

Then $\mathbf{Aut}(S^{\sqcup d} \to S, \mathcal{O}|^r_{S^{\sqcup d}}) \cong (\mathrm{GL}_r)^d \rtimes \mathbb{S}_d$.

$\mathfrak{Mod}_r^{d\text{-ét}}$ is equivalent to the category of $((\mathrm{GL}_r)^d \rtimes \mathbb{S}_d)$–torsors.

Cohomology of Modules

The norm gives a functor $N\colon \mathfrak{Mod}_r^{d\text{-ét}} \to \mathfrak{Mod}_{r^d}$ and

$$N_{S^{\sqcup d}/S}(\mathcal{O}|^r_{S^{\sqcup d}}) = (\mathcal{O}^r)^{\otimes d} = \mathcal{O}^{r^d}.$$

The homomorphism between automorphism groups is the *Segre embedding*

$$\mathrm{Seg}\colon (\mathsf{GL}_r)^d \rtimes \mathbb{S}_d \to \mathsf{GL}_{r^d}$$

which sends $(B_1, \ldots, B_d) \to B_1 \otimes \ldots \otimes B_d$, the tensor of linear maps, and sends $\sigma \in \mathbb{S}_d$ to the permutation of the tensor factors of $\mathcal{O}^{r^d} = (\mathcal{O}^r)^{\otimes d}$.

The induced map on cohomology is

$$\begin{aligned} H^1(S, (\mathsf{GL}_r)^d \rtimes \mathbb{S}_d) &\to H^1(S, \mathsf{GL}_{r^d}) \\ [(T \to S, \mathcal{M})] &\mapsto [N_{T/S}(\mathcal{M})]. \end{aligned}$$

Cohomology of Azumaya Algebras

Similarly, we get a functor $N\colon \mathfrak{Azu}_r^{d\text{-ét}} \to \mathfrak{Azu}_{r^d}$ and

$$N(S^{\sqcup d} \to S, \mathrm{M}_r(\mathcal{O}|_{S^{\sqcup d}})) = \mathrm{M}_{r^d}(\mathcal{O})$$

with associated group homomorphism

$$\mathrm{PSeg}\colon (\mathrm{PGL}_r)^d \rtimes \mathbb{S}_d \to \mathrm{PGL}_{r^d}.$$

Its map on cohomology is

$$\begin{aligned} H^1(S, (\mathrm{PGL}_r)^d \rtimes \mathbb{S}_d) &\to H^1(S, \mathrm{PGL}_{r^d}) \\ [(T \to S, \mathcal{A})] &\mapsto [N_{T/S}(\mathcal{A})]. \end{aligned}$$

Restricting the Segre Embedding

The Segre embedding $(\mathrm{PGL}_{2r})^{2d} \rtimes \mathbb{S}_{2d} \to \mathrm{PGL}_{(2r)^{2d}}$ restricts to

$$(\mathrm{PSp}_{2r})^{2d} \rtimes \mathbb{S}_{d} \to \mathrm{PGO}_{(2r)^{2d}}$$

and in the case $r = 1$, $d = 1$ this yields an isomorphism

$$(\mathrm{PSp}_{2})^{2} \rtimes \mathbb{S}_{2} \xrightarrow{\sim} \mathrm{PGO}_{4}.$$

$A_1 \times A_1 \equiv D_2$

- Type $A_1 \times A_1$:
  - Objects are $(T \to S, \mathcal{Q})$ where $T \to S$ is étale of degree 2 and $\mathcal{Q}$ is a quaternion algebra.
  - Since $\mathrm{PGL}_2 \cong \mathrm{PSp}_2$, these are $((\mathrm{PSp}_2)^2 \rtimes \mathbb{S}_2)$–torsors.
- Type D_2:
  - Objects are $(\mathcal{A}, \sigma, f)$, Azumaya $\mathcal{O}$–algebras of degree 4 with a quadratic pair (σ is an orthogonal involution and $f \colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ is a linear map).
  - Morphisms are $\varphi \colon (\mathcal{A}, \sigma) \xrightarrow{\sim} (\mathcal{A}', \sigma')$ such that $f' \circ \varphi = f$.
  - These are PGO_4–torsors.

$A_1 \times A_1 \equiv D_2$

Theorem

The norm gives an equivalence of categories (more generally of stacks)

$$N\colon A_1 \times A_1 \to D_2$$
$$(T \to S, \mathcal{Q}) \mapsto (N_{T/S}(\mathcal{Q}), \sigma', f').$$

This is a generalization of §15.B in The Book of Involutions where they use the norm for étale extension of a field. A similar result over schemes appeared in a paper by A. Auel where it was assumed $2 \in \mathcal{O}(S)^\times$.

Thank You