Quadratic Pairs over Schemes and the Canonical Quadratic Pair on Clifford Algebras

Cameron Ruether

IMAR București

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Usual Clifford Algebras

Let $q\colon V \to \mathbb{F}$ be a non-singular quadratic form, $\dim(V) = 2n$, $\operatorname{char}(\mathbb{F}) \neq 2$.

$$\mathrm{Cl}(V, q) = T(V)/\langle v \otimes v - q(v) \cdot 1 \mid v \in V \rangle.$$

Canonical involution $\underline{\sigma}\colon v_1 \otimes \ldots \otimes v_k \mapsto v_k \otimes \ldots \otimes v_1$.

$$\mathrm{Cl}_0(V, q) = \mathrm{Span}_{\mathbb{F}}(\{v_1 \otimes \ldots \otimes v_k \mid k \text{ even}\}) \subset \mathrm{Cl}(V, q)$$

$\underline{\sigma}_0 = \underline{\sigma}|_{\mathrm{Cl}_0(V,q)}$ also called canonical involution.

$$\underline{\sigma}_0 \text{ is } \begin{cases} \text{unitary} & n \text{ odd} \\ \text{symplectic} & n \equiv 2 \pmod 4 \\ \text{orthogonal} & n \equiv 0 \pmod 4. \end{cases}$$

$$\mathbf{O}(V, q) \to \mathbf{PGO}(\mathrm{Cl}_0(V, q), \underline{\sigma}_0)$$
$$\varphi \mapsto (v_1 \otimes v_2 \mapsto \varphi(v_1) \otimes \varphi(v_2))$$

Usual Clifford Algebras

Let (A, σ) be a central simple $\mathbb{F}$–algebra, $\deg(A) = 2n$, σ orthogonal.

$$\mathrm{Cl}(A, \sigma) = \frac{T(A)}{J_1 + J_2}$$

	(A, σ)	$(\mathrm{End}_{\mathbb{F}}(V), \sigma_q) \cong (V \otimes_{\mathbb{F}} V, \mathrm{sw})$
J_1	$\langle a - \frac{1}{2}\,\mathrm{Trd}_A(a) \mid \sigma(a) = a \rangle$	$\langle v \otimes v - q(v) \cdot 1 \rangle$
J_2	complicated	$\langle (w_1 \otimes v) \otimes (v \otimes w_2) - q(v) w_1 \otimes w_2 \rangle$

Canonical involution $\underline{\sigma}(a_1 \otimes \ldots \otimes a_k) = \sigma(a_k) \otimes \ldots \otimes \sigma(a_1)$.

$$\underline{\sigma} \text{ is } \begin{cases} \text{unitary} & n \text{ odd} \\ \text{symplectic} & n \equiv 2 \pmod 4 \\ \text{orthogonal} & n \equiv 0 \pmod 4. \end{cases}$$

$$\mathbf{PGO}(A, \sigma) \to \mathbf{PGO}(\mathrm{Cl}(A, \sigma), \underline{\sigma})$$
$$\varphi \mapsto (a \mapsto \varphi(a)).$$

Conventions

Over arbitrary $\mathbb{F}$ (later over a scheme S). Let

$$\sigma\colon A \to A$$

be an involution of the first kind, split A

$$\to \sigma'\colon \operatorname{End}_{\mathbb{F}'}(V) \to \operatorname{End}_{\mathbb{F}'}(V)$$

adjoint to a regular bilinear form $b_\sigma\colon V \times V \to \mathbb{F}'$.

$$\text{call } \sigma \begin{cases} \text{orthogonal} & \text{if } b_\sigma \text{ symmetric} \\ \text{weakly-symplectic} & \text{if } b_\sigma \text{ skew-symmetric} \\ \text{symplectic} & \text{if } b_\sigma \text{ alternating.} \end{cases}$$

Quadratic Pairs

$\mathbb{F}$ an arbitrary field, A central simple $\mathbb{F}$-algebra.

Definition [KMRT]

A *quadratic pair* on A is (σ, f) where

- σ is an orthogonal involution,
- $f\colon \mathrm{Sym}(A,\sigma) \to \mathbb{F}$ is linear such that

$$f(a+\sigma(a)) = \mathrm{Trd}_A(a) \quad \forall a \in A.$$

If $\mathrm{char}(\mathbb{F}) \neq 2$, $s \in \mathrm{Sym}(A,\sigma)$

$$f(s) = f\left(\frac{1}{2}(s+\sigma(s))\right) = \frac{1}{2}f(s+\sigma(s)) = \frac{1}{2}\,\mathrm{Trd}_A(s).$$

Connection to Quadratic Forms

Theorem [KMRT]

Consider $(\mathrm{End}_{\mathbb{F}}(V), \sigma_b)$ for a regular symmetric bilinear $b\colon V \times V \to \mathbb{F}$. We have $\varphi_b\colon (V \otimes_{\mathbb{F}} V, \mathrm{sw}) \to (\mathrm{End}_{\mathbb{F}}(V), \sigma_b)$. There is a bijection

$$\left\{ \begin{array}{c} f \text{ such that } (\sigma_b, f) \\ \text{is a quadratic pair} \end{array} \right\} \leftrightarrow \{q\colon V \to \mathbb{F} \text{ with polar } b\}$$

$$f \mapsto q_f(v) = f(\varphi_b(v \otimes v))$$

$$f_q(\varphi_b(v \otimes v)) = q(v) \leftarrow q$$

$$f_q(\varphi_b(v \otimes w + w \otimes v)) = b_q(v, w)$$

Classification [KMRT]

For all (A, σ, f) there exists $\ell \in A$ such that

- $\ell + \sigma(\ell) = 1$
- $f(\_\_) = \mathrm{Trd}_A(\ell \cdot \_\_)$
- ℓ is unique up to adding an element of $\mathrm{Alt}(A, \sigma) = \{a - \sigma(a) \mid a \in A\}$.

Conversely, given (A, σ), any $\ell \in A$ with $\ell + \sigma(\ell) = 1$ defines

$$f_\ell = \mathrm{Trd}_A(\ell \_\_)$$

and (σ, f_ℓ) is a quadratic pair. $f_\ell = f_{\ell'} \Leftrightarrow \ell - \ell' \in \mathrm{Alt}(A, \sigma)$.

More Clifford Algebras

Let (A, σ, f) be an $\mathbb{F}$-c.s.a. with quadratic pair, $\deg(A) = 2n$.

$$\mathrm{Cl}(A, \sigma, f) = \frac{T(A)}{J_1 + J_2}$$

	(A, σ, f)	$(\mathrm{End}_{\mathbb{F}}(V), \sigma, f)$, $\varphi\colon V \otimes_{\mathbb{F}} V \xrightarrow{\sim} \mathrm{End}_{\mathbb{F}}(V)$
J_1	$\langle a - f(a) \mid \sigma(a) = a \rangle$	$\langle v \otimes v - f(\varphi(v \otimes v)) \cdot 1 \rangle$
J_2	still complicated	$\langle (w_1 \otimes v) \otimes (v \otimes w_2) - f(\varphi(v \otimes v)) w_1 \otimes w_2 \rangle$

Canonical involution $\underline{\sigma}(a_1 \otimes \ldots \otimes a_k) = \sigma(a_k) \otimes \ldots \otimes \sigma(a_1)$.

$$\underline{\sigma} \text{ is orthogonal} \Leftrightarrow \begin{array}{ll} n \equiv 0 \pmod 4, \text{ or} \\ n \equiv 2 \pmod 4 \text{ and } \mathrm{char}(\mathbb{F}) = 2. \end{array}$$

For clarity, $c\colon A \to \mathrm{Cl}(A, \sigma, f)$ is the map $a \mapsto c(a) = a$.

Canonical Quadratic Pair

When $\underline{\sigma}$ is orthogonal, $(\mathrm{Cl}(A,\sigma,f), \underline{\sigma}, ?)$.

Want: $\mathbf{PGO}(A,\sigma,f) \longrightarrow \mathbf{Aut}(\mathrm{Cl}(A,\sigma,f),\underline{\sigma})$

$\mathbf{PGO}(A,\sigma,f) \rightarrow \mathbf{PGO}(\mathrm{Cl}(A,\sigma,f),\underline{\sigma}, ?) \hookrightarrow \mathbf{Aut}(\mathrm{Cl}(A,\sigma,f),\underline{\sigma})$

Theorem (Dolphin, Quéguiner-Mathieu (2021))

Let $\deg(A) \geq 8$ *such that* $\underline{\sigma}$ *is orthogonal. Take any* $a \in A$ *with* $\mathrm{Trd}_A(a) = 1$ *and use* $\ell = c(a) \in \mathrm{Cl}(A,\sigma,f)$*. Then,*

$$\underline{f} = \mathrm{Trd}_{\mathrm{Cl}(A,\sigma,f)}(c(a)\cdot \_\_)$$

- *does not depend on the choice of* $a \in A$*, and*
- $\underline{f}$ *fills* ? *in the diagram above.*

Canonical Quadratic Pair

$$\begin{aligned}
\ell + \underline{\sigma}(\ell) &= c(a) + \underline{\sigma}(c(a)) \\
&= c(a) + c(\sigma(a)) \\
&= c(a + \sigma(a)) \\
&= f(a + \sigma(a)) \cdot 1 && \text{due to } J_1 \\
&= \mathrm{Trd}_A(a) \cdot 1 && \text{by definition of } f \\
&= 1 && \text{by assumption.}
\end{aligned}$$

Over a Scheme

Let S be an arbitrary base scheme. $(\mathfrak{Sch}_S, \mathcal{O})$ the ringed fppf-site over S. $(\mathcal{A}, \sigma)$ is an Azumaya algebra of constant degree $2n$ with orthogonal involution.

- $\mathcal{A}\colon \mathfrak{Sch}_S \to \mathfrak{Ab}$ is a sheaf, in particular an $\mathcal{O}$–module, locally isomorphic to $\mathrm{M}_{2n}(\mathcal{O})$.
- $\sigma\colon \mathcal{A} \to \mathcal{A}$ is a natural transformation, locally adjoint to symmetric bilinear forms.

$$\begin{aligned} \mathcal{S}ym_{\mathcal{A},\sigma} &= \ker(\mathrm{Id} - \sigma) \subset \mathcal{A} \\ \mathcal{A}\ell t_{\mathcal{A},\sigma} &= \mathrm{Img}(\mathrm{Id} - \sigma) \subset \mathcal{A} \\ \mathcal{S}ymd_{\mathcal{A},\sigma} &= \mathrm{Img}(\mathrm{Id} + \sigma) \subset \mathcal{A}. \end{aligned}$$

Over a Scheme

Definition [Calmès, Fasel]

A *quadratic pair* on $\mathcal{A}$ is (σ, f) where

- σ is an orthogonal involution, and
- $f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ is $\mathcal{O}$–linear such that

$$f(a+\sigma(a)) = \mathrm{Trd}_A(a) \quad \forall T \in \mathfrak{Sch}_S,\ \forall a \in \mathcal{A}(T).$$

For $(\mathcal{A}, \sigma, f)$:

$$\mathcal{C}\ell(\mathcal{A}, \sigma, f) = \frac{\mathcal{T}(\mathcal{A})}{\mathcal{J}_1 + \mathcal{J}_2}$$

$c\colon \mathcal{A} \to \mathcal{C}\ell(\mathcal{A}, \sigma, f)$, canonical involution $\underline{\sigma}\colon c(a) \mapsto c(\sigma(a))$, and

$$\underline{\sigma} \text{ is orthogonal} \Leftrightarrow \begin{array}{ll} n \equiv 0 & \pmod 4, \text{ or} \\ n \equiv 2 & \pmod 4 \text{ and } 2 = 0 \in \mathcal{O}(S). \end{array}$$

Problem 1

Given $(\mathcal{A}, \sigma, f)$, there may not exist $\ell \in \mathcal{A}(S)$ such that $f(\_\_) = \mathrm{Trd}_{\mathcal{A}}(\ell \cdot \_\_)$.

Example [Gille, Neher, R.]

$S = E$ an ordinary elliptic curve over $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) = 2$. Consider the $(\mu_2 \times_{\mathbb{F}} \mathbb{Z}/2\mathbb{Z})$–torsor $E' = E \xrightarrow{\cdot 2} E$. Then,

$$(Q, \sigma) = E' \wedge^{\mu_2 \times_{\mathbb{F}} \mathbb{Z}/2\mathbb{Z}} (\mathrm{M}_2(\mathcal{O}), \psi)$$

is a quaternion algebra over E.

- There exists f such that $(\mathcal{Q}, \sigma, f)$ is a quadratic pair.
- $Q(E) \cong \mathbb{F}$ and so $Q(E) = \mathcal{S}ym_{\mathcal{Q},\sigma}(E)$.
- Then $\ell + \sigma(\ell) = 2\ell = 0$ for all $\ell \in \mathcal{Q}(E)$.

Solution 1

Lemma [GNR]

If $(\mathcal{A}, \sigma, f)$ is an Azumaya algebra with quadratic pair, then $1_{\mathcal{A}} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$.

Given a *locally quadratic* $(\mathcal{A}, \sigma)$ (i.e., with $1 \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$),

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}+\sigma} & \mathcal{S}ymd_{\mathcal{A},\sigma} \\ \downarrow & & \| \\ \mathcal{A}/\mathcal{A}\ell t_{\mathcal{A},\sigma} & \xrightarrow{\xi} & \mathcal{S}ymd_{\mathcal{A},\sigma} \end{array}$$

Theorem [GNR]

There is a bijection of sets

$$\left\{ \begin{array}{c} f \text{ such that } (\mathcal{A}, \sigma, f) \\ \text{is a quadratic pair on } \mathcal{A} \end{array} \right\} \leftrightarrow \xi(S)^{-1}(1_{\mathcal{A}}) \subset (\mathcal{A}/\mathcal{A}\ell t_{\mathcal{A},\sigma})(S).$$

Solution 1

Lemma [GNR]

Given $(\mathcal{A}, \sigma, f)$, if S is affine there exists $\ell \in \mathcal{A}(S)$ such that $f = \mathrm{Trd}_{\mathcal{A}}(\ell \cdot \_\_)$.

Fix $(\mathcal{A}, \sigma)$. Take affine cover $\{U_i \to S\}_{i \in I}$. Given f,

$$\begin{aligned} &\to \ell_i \in \mathcal{A}(U_i) \\ &\to [\ell_i] \in (\mathcal{A}/\mathcal{A}\ell t_{\mathcal{A},\sigma})(U_i), \text{ these glue} \\ &\Rightarrow [\ell] \in (\mathcal{A}/\mathcal{A}\ell t_{\mathcal{A},\sigma})(S) \\ &\Rightarrow \pi([\ell]) = 1 \text{ since } \ell_i + \sigma(\ell_i) = 1. \end{aligned}$$

Given $[\ell] \in (\mathcal{A}/\mathcal{A}\ell t_{\mathcal{A},\sigma})(S)$ with $\pi([\ell]) = 1$,

$$\begin{aligned} &\to \text{ get } \ell_i \in \mathcal{A}(U_i) \text{ with } \ell_i + \sigma(\ell_i) = 1, \\ &\to f_i = \mathrm{Trd}_{\mathcal{A}|_{U_i}}(\ell_i \cdot \_\_) \colon \mathcal{S}ym_{\mathcal{A},\sigma}|_{U_i} \to \mathcal{O}|_{U_i}, \text{ these glue} \\ &\Rightarrow f \colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}. \end{aligned}$$

Problem 2

Given $(\mathcal{A}, \sigma, f)$, there may not exist $a \in \mathcal{A}(S)$ with $\mathrm{Trd}_{\mathcal{A}}(a) = 1$.

Example [GNR]

Take $\mathcal{Q}$ as above. $\mathcal{Q}(E) = \mathcal{O}(E) \cong \mathbb{F}$. Whenever $\mathcal{Q}(T) \cong \mathrm{M}_2(\mathcal{O}(T))$,

$$\mathcal{Q}(E) \to \mathrm{M}_2(\mathcal{O}(T))$$
$$c \mapsto \begin{bmatrix} c|_T & 0 \\ 0 & c|_T \end{bmatrix}$$

So $\mathrm{Trd}_{\mathcal{A}}(c) = 0$.

Solution 2

$\mathrm{Trd}_{\mathcal{A}} \colon \mathcal{A} \twoheadrightarrow \mathcal{O}$ is a surjective map of sheaves. $\ker(\mathrm{Trd}_{\mathcal{A}}) = \mathfrak{sl}_{\mathcal{A}}$.

Lemma

If $\deg(\mathcal{A}) \geq 6$ there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{sl}_{\mathcal{A}} & \longrightarrow & \mathcal{A} & \xrightarrow{\mathrm{Trd}_{\mathcal{A}}} & \mathcal{O} & \longrightarrow & 0 \\
 & & \downarrow c & & \downarrow c & & \downarrow \exists\rho & & \\
0 & \longrightarrow & \mathcal{A}lt_{\mathcal{C}\ell,\underline{\sigma}} & \longrightarrow & \mathcal{C}\ell(\mathcal{A},\sigma,f) & \longrightarrow & \mathcal{C}\ell(\mathcal{A},\sigma,f)/\mathcal{A}lt_{\mathcal{C}\ell,\underline{\sigma}} & \longrightarrow & 0 \\
 & & & & \downarrow \mathrm{Id}+\underline{\sigma} & & \downarrow \pi & & \\
 & & & & \mathcal{S}ymd_{\mathcal{C}\ell,\underline{\sigma}} & = & \mathcal{S}ymd_{\mathcal{C}\ell,\underline{\sigma}} & &
\end{array}$$

and

$$\begin{array}{ccccc}
\mathcal{O} & \xrightarrow{\rho} & \mathcal{C}\ell(\mathcal{A},\sigma,f)/\mathcal{A}lt_{\mathcal{C}\ell,\underline{\sigma}} & \xrightarrow{\pi} & \mathcal{S}ymd_{\mathcal{C}\ell,\underline{\sigma}} \\
a & \longmapsto & & & a \cdot 1_{\mathcal{C}\ell}
\end{array}$$

The Canonical Pair deg ≥ 8

$\rho\colon \mathcal{O} \to \mathcal{C}\ell(\mathcal{A}, \sigma, f)/\mathcal{A}\ell t_{\mathcal{C}\ell, \underline{\sigma}}$

Definition

Let $(\mathcal{A}, \sigma, f)$ be such that $(\mathcal{C}\ell(\mathcal{A}, \sigma, f), \underline{\sigma})$ is orthogonal, $\deg(\mathcal{A}) \geq 8$. Call the semi-trace $\underline{f}$ corresponding to $\rho(1)$ the canonical semi-trace.

Theorem

We have a commutative diagram

$$\mathbf{PGO}(\mathcal{A}, \sigma, f) \longrightarrow \mathbf{Aut}(\mathcal{C}\ell(\mathcal{A}, \sigma, f), \underline{\sigma})$$

$$\mathbf{PGO}(\mathcal{A}, \sigma, f) \longrightarrow \mathbf{PGO}(\mathcal{C}\ell(\mathcal{A}, \sigma, f), \underline{\sigma}, \underline{f}) \hookrightarrow \mathbf{Aut}(\mathcal{C}\ell(\mathcal{A}, \sigma, f), \underline{\sigma})$$

If $S = \mathrm{Spec}(\mathbb{F})$ this recovers [DQ21].

Twisting

Let $\mathcal{V} = \mathrm{Span}_{\mathcal{O}}(\{v_1, \ldots, v_n\})$, $\mathbb{H}(\mathcal{V}) = \mathrm{Span}_{\mathcal{O}}(\{v_1, \ldots, v_n, v_n^*, \ldots, v_1^*\})$ with hyperbolic quadratic form

$$q_{2n}(a_1 v_1 + \ldots a_n v_n + a_n^* v_n^* + \ldots + a_1^* v_1^*) = \sum_{i=1}^{n} a_i a_i^*$$

$\to (\mathcal{E}nd_{\mathcal{O}}(\mathbb{H}(\mathcal{V})), \sigma_{2n}, f_{2n})$
$\to \mathcal{C}_0 = (\mathcal{C}\ell(\mathcal{E}nd_{\mathcal{O}}(\mathbb{H}(\mathcal{V})), \sigma_{2n}, f_{2n}), \underline{\sigma_{2n}}, \underline{f_{2n}})$.

For $(\mathcal{A}, \sigma, f)$ of degree $2n$, let $\mathcal{P} = \mathcal{I}som((\mathcal{E}nd_{\mathcal{O}}(\mathbb{H}(\mathcal{V})), \sigma_{2n}, f_{2n}), (\mathcal{A}, \sigma, f))$

$$(\mathcal{C}\ell(\mathcal{A}, \sigma, f), \underline{\sigma}, \underline{f}) \cong \mathcal{P} \wedge^{\mathbf{PGO}_{2n}} \mathcal{C}_0$$

No Canonical Pair when $\deg = 4$

Only excluded case when $\underline{\sigma}$ is orthogonal is $\deg(\mathcal{A}) = 4$ and $2 = 0 \in \mathcal{O}(S)$.

Theorem

Let $(\mathcal{A}, \sigma, f)$ be degree 4 and $2 = 0 \in \mathcal{O}(S)$. Then there is no canonical semi-trace on $(\mathcal{C}\ell(\mathcal{A}, \sigma, f), \underline{\sigma})$. In particular, for all $f' \colon \mathcal{S}ym_{\mathcal{C}\ell, \underline{\sigma}} \to \mathcal{O}$,

$$\begin{array}{ccc} \mathbf{PGO}(\mathcal{A}, \sigma, f) & \xrightarrow{\qquad\qquad\qquad} & \mathbf{Aut}(\mathcal{C}\ell(\mathcal{A}, \sigma, f), \underline{\sigma}) \\ & \overset{\nexists}{\searrow} \qquad \nearrow & \\ & \mathbf{PGO}(\mathcal{C}\ell(\mathcal{A}, \sigma, f), \underline{\sigma}, f') & \end{array}$$

Problem: $\mathcal{A}\ell t_{\mathcal{C}\ell(\mathcal{E}nd_{\mathcal{O}}(\mathbb{H}(\mathcal{V}))), \underline{\sigma_4}}$ is too small.

Problem 3

Assume S affine. For $(\mathcal{C}\ell(\mathcal{E}nd_{\mathcal{O}}(\mathbb{H}(\mathcal{V})), \sigma_4, f_4), \underline{\sigma_4}, f')$, $f' = \mathrm{Trd}_{\mathcal{C}\ell}(\ell \cdot \_\_)$.
$\varphi \in \mathbf{PGO}(\mathcal{E}nd_{\mathcal{O}}(\mathbb{H}(\mathcal{V})), \sigma_4, f_4)$ respects f' if and only if $\ell - \varphi(\ell) \in \mathcal{A}\ell t_{\mathcal{C}\ell, \underline{\sigma_4}}$.

$$\Rightarrow \ell = 1 + e_1 e_1^* + e_2 e_2^*$$

but this is only fixed if $t^2 = t$ for all $t \in \mathcal{O}(S)$.

Example

For a boolean ring R and $(\mathrm{M}_2(R), \sigma_4, f_4)$, there exists $\ell \in \mathrm{Cl}(\mathrm{M}_2(R), \sigma_4, f_4)$ such that

$$f' = \mathrm{Trd}_{\mathcal{C}\ell}(\ell \cdot \_\_)$$

is a semi-trace for $\underline{\sigma_4}$ fixed by all $\varphi \in \mathbf{PGO}_4(R)$.

Thank You