Killing Forms, W-Invariants, and the Tensor Product Map

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Context

- Interested in invariant quadratic forms associated to linear algebraic groups.
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S. Garibaldi, A. Merkurjev, J.-P. Serre construct $Q(G)$ in [1].

$Q(G)$ Appears in work by S. Garibaldi [2], S. Baek [3], as well as by A. Merkurjev, A. Neshitov, and K. Zainoulline [4] relating to cohomological invariants of linear algebraic groups.
Let $G$ be a split, semisimple, linear algebraic group (over an alg. closed field $\mathbb{F}$) with a maximal torus $T$. 

$G$ has a root system $\Phi \subseteq T^*$ with Weyl group $W$. $W$ acts on $\Phi$ by permuting its elements, but since $W$ is crystallographic this action extends to all of $T^*$. In particular $W$ acts on the symmetric tensor product $S(T^*)$, and so we can discuss invariant quadratic forms. 

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Linear Algebraic Group
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An example of a fixed element is the *Killing form* \( K = \sum_{\alpha \in \Phi} \alpha^2 \).
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Analogous to the Killing form in Lie theory, $\mathcal{K}(x, y) = \text{Tr}(\text{ad}(x) \text{ad}(y))$ on $\text{Lie}(G)$. 
When $G$ is a simple group, $S^2(\mathcal{T}^*)^W = \mathbb{Z}\langle q \rangle$ where $q$ is called the normalized Killing form.
W-Invariants

- When $G$ is a simple group, $S^2(T^*)^W = \mathbb{Z}\langle q \rangle$ where $q$ is called the normalized Killing form.
- If $G$ is semisimple, $S^2(T^*)^W = \mathbb{Z}\langle q_1 \rangle \oplus \ldots \oplus \mathbb{Z}\langle q_m \rangle$. 
### Examples

<table>
<thead>
<tr>
<th>Group</th>
<th>Killing Form</th>
<th>Normalized Killing Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SL}(V), \dim(V) = n + 1$</td>
<td>$4(n + 1) \sum_{i,j=1}^{n} e_i e_j$</td>
<td>$\sum_{i,j=1}^{n} e_i e_j$</td>
</tr>
<tr>
<td>$\text{SO}(V), \dim(V) = 2n$</td>
<td>$4(n - 1) \sum_{i=1}^{n} e_i^2$</td>
<td>$\sum_{i=1}^{n} e_i^2$</td>
</tr>
<tr>
<td>$\text{SO}(V), \dim(V) = 2n + 1$</td>
<td>$4(n - 2) \sum_{i=1}^{n} e_i^2$</td>
<td>$\sum_{i=1}^{n} e_i^2$</td>
</tr>
<tr>
<td>$\text{Sp}(V), \dim(V) = 2n$</td>
<td>$4(n + 1) \sum_{i=1}^{n} e_i^2$</td>
<td>$\sum_{i=1}^{n} e_i^2$</td>
</tr>
</tbody>
</table>

Where in call cases $T^* = \langle e_i \mid 1 \leq i \leq n \rangle$. 

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Induced Map on W-Invariants

- $Q(G)$ is functorial. If $\rho: G \to H$ is a homomorphism we have

$$\rho^*: S^2(T^*_H)^W \to S^2(T^*_G)^W$$
Induced Map on W-Invariants

- $Q(G)$ is functorial. If $\rho : G \rightarrow H$ is a homomorphism we have

$$\rho^* : S^2(T_H^*)^W \rightarrow S^2(T_G^*)^W$$

- Since $S^2(T_H^*)^W$ is generated by some normalized Killing forms $q_1, \ldots, q_m$, this map is described by their images, called the Rost multipliers of $\rho$. 
The tensor product map

\[ \rho: \text{GL}(V_1) \times \text{GL}(V_2) \to \text{GL}(V_1 \otimes V_2) \]

\[ (A, B) \mapsto A \otimes B \]
The tensor product map

\[ \rho : \text{GL}(V_1) \times \text{GL}(V_2) \to \text{GL}(V_1 \otimes V_2) \]

\[(A, B) \mapsto A \otimes B\]

If \(A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix},\) then \(A \otimes B = \begin{bmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ldots & a_{nn}B \end{bmatrix}.\)
Kroenecker Tensor Product Map

- The tensor product map

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- If \( A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \), \( A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \).

- In general we consider

\[ \rho: \text{GL}(V_1) \times \cdots \times \text{GL}(V_m) \to \text{GL}(V_1 \otimes \cdots \otimes V_m) \]

\[(A_1, \ldots, A_m) \mapsto A_1 \otimes \cdots \otimes A_m\]
Consider restrictions of this map to the special linear, special orthogonal, and symplectic groups.
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For the following cases

- $G_1, \ldots, G_n, H = \text{SL}$
- $G_1, \ldots, G_{2m} = \text{Sp}$
  $G_{2m+1}, \ldots, G_n = \text{SO}$
  $H = \text{SO}$
- $G_1, \ldots, G_{2m+1} = \text{Sp}$
  $G_{2m+2}, \ldots, G_n = \text{SO}$
  $H = \text{Sp}$

we consider

$$\rho : G_1(V_1) \times \ldots \times G_n(V_n) \to H(V_1 \otimes \ldots \otimes V_n)$$
Example Computation of $\rho^*$

$$\rho: \text{SO}(\mathbb{F}^{2n+1}) \times \text{SO}(\mathbb{F}^{2m+1}) \rightarrow \text{SO}(\mathbb{F}^{(2n+1)(2m+1)})$$
Example Computation of $\rho^*$

\[ \rho: \text{SO}(\mathbb{F}^{2n+1}) \times \text{SO}(\mathbb{F}^{2m+1}) \to \text{SO}(\mathbb{F}^{(2n+1)(2m+1)}) \]

Choose \( T_{2n+1} = \{ \text{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i \in \mathbb{F}^\times \} \)

and others similarly.
Example Computation of $\rho^*$

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- Choose $T_{2n+1} = \{\text{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}$ and others similarly.
- $T^*_{2n+1} = \langle e_i \mid 1 \leq i \leq n \rangle$ where $e_i(\text{diag}(t_1, \ldots, t_1^{-1})) = t_i$.
- $T^*_{(2n+1)(2m+1)} = \langle f_i \mid 1 \leq i \leq 2nm + n + m \rangle$
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- Choose $T_{2n+1} = \{\text{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i \in \mathbb{F}^\times\}$ and others similarly.
- $T_{2n+1}^* = \langle e_i \mid 1 \leq i \leq n \rangle$ where $e_i(\text{diag}(t_1, \ldots, t_1^{-1})) = t_i$.
- $T_{(2n+1)(2m+1)}^* = \langle f_i \mid 1 \leq i \leq 2nm + n + m \rangle$

\[
\rho^*(f_i) = \begin{cases} 
(e_{k+1}, e_r) & 0 \leq k \leq n - 1, 1 \leq r \leq m \\
(e_{k+1}, 0) & r = m + 1 \\
(e_{k+1}, -e_{2m+2-r}) & m + 2 \leq r \leq 2m + 1, k = n \\
(0, e_r) & 1 \leq r \leq m 
\end{cases}
\]

where $i = k(2m + 1) + r$ with $0 \leq k \leq 2n$ and $1 \leq r \leq 2m + 1$. 
Example Computation of $\rho^*$

\[
\rho^*\left(q(2n+1)(2m+1)\right) = \rho^* \left(\sum_{i=1}^{2nm+n+m} f_i^2\right) = \sum_{i=1}^{2nm+n+m} \rho^* (f_i)^2.
\]
Example Computation of $\rho^*$

\[ \rho^*(q_{(2n+1)(2m+1)}) = \rho^* \left( \sum_{i=1}^{2nm+n+m} f_i^2 \right) = \sum_{i=1}^{2nm+n+m} \rho^*(f_i)^2. \]
Example Computation of $\rho^*$

1. $\rho^*(q_{(2n+1)(2m+1)}) = \rho^* \left( \sum_{i=1}^{2nm+n+m} f_i^2 \right) = \sum_{i=1}^{2nm+n+m} \rho^*(f_i)^2$.

2. 

3. $((2m + 1)q_{2n+1}, (2n + 1)q_{2m+1})$. 
Results

Theorem
Let $V_1, \ldots, V_n$ be vector spaces such that $\dim(V_i) = d_i$. Consider linear algebraic groups $G_1, \ldots, G_n, H$ in one of the previous configurations (where $G_i = \text{Sp}$ only when $d_i$ is even). Consider the Kronecker product map

$$\rho: G_1(V_1) \times \ldots \times G_n(V_n) \to H(V_1 \otimes \ldots \otimes V_n)$$

and let $q_1, \ldots, q_n, q_H$ be the respective normalized Killing forms. Then

$$\rho|_{n}^*(q_H) = \left( (d_2 \ldots d_n)q_1, \ldots, (d_1 \ldots \hat{d}_i \ldots d_n)q_i, \ldots, (d_1 \ldots d_{n-1})q_n \right)$$

where $\hat{d}_i$ represents omission.
Thank You.

