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My research interests lie within algebraic geometry where I focus on linear algebraic group schemes, their cohomology, and related objects. In my Ph.D. research, I studied linear algebraic groups defined over a field of characteristic not 2 by examining their Chevalley generators and Galois cohomology. This allowed me to prove new results about the cohomological invariants of the half–spin groups. This work involved studying Clifford algebras, spin groups, and other objects related to quadratic forms, which, when defined over the real numbers, appear often in physics. Now, in my current research, I am applying techniques from my Ph.D. to study linear algebraic groups defined over a general base scheme. This new setting allows for many more possibilities, for example it includes cases where the characteristic is 2, which complicates the behaviour of groups such as the special orthogonal or spin groups. In a joint research project with Erhard Neher and Philippe Gille, we have already developed new cohomological invariants over schemes related to quadratic pairs, which are generalizations of orthogonal involutions. These invariants demonstrate behaviour unique to general schemes which does not occur over rings or fields. Furthermore, building off the work in this joint project, I have independently generalized the construction over fields of the canonical quadratic pair on Clifford algebras to the setting over a scheme. With this key piece in hand, it will be possible to develop the theory of triality over a scheme.

The main tools in the setting over a scheme are fppf<sup>1</sup> descent and fppf cohomology, which are, loosely, the study of how objects can be reconstructed from smaller pieces called localizations. I am interested in the computational properties of working fppf locally, in particular with respect to Chevalley generators. Chevalley generators provide an explicit presentation of linear algebraic groups in many cases which allows one to perform calculations. It is not true in general that the Chevalley generators of a linear algebraic group generate the full group, but on a sufficiently small local piece of the base scheme the generators do produce the full group. Therefore, computations with these generators, together with some descent considerations, can be used to explicitly understand the linear algebraic group in this setting. Furthermore, many of the objects I study, such as the groups themselves or Azumaya algebras with quadratic pairs, form stacks. Stacks are objects which formalize concepts of descent, and as such include a wealth cohomological information. I am interested in the interplay between cohomological statements and stack theoretic statements. For example, many results in the literature which are stated purely cohomologically end up coming from a stack morphism which was in the background and, conversely, constructing new stack morphisms produces new cohomological statements.

My recent work focuses on studying groups of type D over a scheme, i.e., groups of orthogonal type related to quadratic forms, by using the techniques above. My current focus is on describing triality in this setting, as mentioned above. The Dynkin diagrams of type  $D_n$  and type  $D_4$  are depicted below, respectively.



When n > 4, the only symmetry of the diagram  $D_n$  is the horizontal reflection which exchanges the rightmost two vertices. However, the diagram  $D_4$  has additional rotational symmetries. Triality refers to the phenomena which arise from these extra symmetries. These phenomena have been described in settings over a field, but have not yet been studied over a scheme. I also have my eye towards continuing my study of the half-spin groups. As of now, there is no concise description of the half-spin groups. Instead, they are defined as scheme theoretic images under the related half-spin representation. Describing triality will make heavy use of the half-spin representation for  $D_4$  and I plan to generalize techniques used for studying the representation in this case to study it for other  $D_n$  as well.

<sup>&</sup>lt;sup>1</sup>fidèlement plate de présentation finie = faithfully flat and finitely presented.

## 1. QUADRATIC PAIRS, TRIALITY, AND HALF-SPIN

The following sections elaborate on my recent work with Erhard Neher and Philippe Gille, [GNR], where we studied the cohomological properties of quadratic pairs over a scheme, as well as my generalization in [Rue23] of a construction of Dolphin and Quéguiner-Mathieu to define the canonical quadratic pair on a Clifford algebra over a scheme.

1.A. Quadratic Pairs. Groups of type D are classically studied over a field  $\mathbb{F}$  of characteristic not 2. In this context, they are related to central simple algebras with orthogonal involutions. To study these groups over fields of characteristic 2 as well, the authors of The Book of Involutions, [KMRT], introduce quadratic pairs. A quadratic pair  $(\sigma, f)$  on a central simple  $\mathbb{F}$ -algebra A is

- (i) an orthogonal involution  $\sigma$  on A, and
- (ii) a linear map  $f: \text{Sym}(A, \sigma) \to \mathbb{F}$ , where  $\text{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\}$  is the set of symmetric elements, such that

$$f(a + \sigma(a)) = \operatorname{Trd}_A(a)$$

for all  $a \in A$ . Here,  $\operatorname{Trd}_A$  denotes the reduced trace of the central simple algebra. Such a linear map f is called a *semi-trace*.

Quadratic pairs are a suitable replacement for orthogonal involutions in characteristic 2. For example, the group of algebra automorphisms which commute with  $\sigma$ ,  $\operatorname{Aut}(A, \sigma)$ , may not be a smooth group in characteristic 2. However, the group  $\operatorname{Aut}(A, \sigma, f)$  of those algebra automorphisms  $\varphi \colon A \to A$  such that  $\varphi \circ \sigma = \sigma \circ \varphi$  and  $f \circ \varphi = f$  does give rise to a smooth semisimple adjoint group of type D.

The study of quadratic pairs was generalized from fields to schemes by Calmès and Fasel in [CF]. In this setting, working over a fixed base scheme S, most objects are sheaves on the ringed site  $(\mathfrak{Sch}_S, \mathcal{O})$ , where  $\mathfrak{Sch}_S$  is given then fppf topology and  $\mathcal{O}: \mathfrak{Sch}_S \to \mathfrak{Rings}$  is the global sections functor. For example, an *Azumaya algebra* is a sheaf of  $\mathcal{O}$ -modules  $\mathcal{A}$  for which there exists an fppf cover of S, say  $\{T_i \to S\}_{i \in I}$ , over which  $\mathcal{A}|_{T_i} \cong \mathfrak{M}_{n_i}(\mathcal{O}|_{T_i})$  for some  $n_i \in \mathbb{N}$  for each  $i \in I$ . In the main results of [GNR], we provide various cohomological obstructions which detect phenomenon such as the non-existence of suitable semi-traces f, when an f extends from  $\mathcal{Sym}_{\mathcal{A},\sigma}$  to all of  $\mathcal{A}$ , and others. We also provide a classification of the possible semi-traces f which may extend a given Azumaya algebra with orthogonal involution  $(\mathcal{A}, \sigma)$  into an algebra with quadratic pair  $(\mathcal{A}, \sigma, f)$ . This classification differs subtly from the classification over fields.

The theory of triality, which relies heavily on Clifford algebras, is explained in the final chapters of [KMRT] only over fields of characteristic not 2. This assumption, uncharacteristic for [KMRT], was added because the authors were not able to describe a suitable canonical quadratic pair on Clifford algebras. However, in the setting over a field of characteristic 2, recent work by Dolphin and Quéguiner-Mathieu, [DQ], described the appropriate canonical quadratic pair on the Clifford algebra. Using results from [GNR], I was able to generalize their construction and provide a definition of the canonical quadratic pair on a Clifford algebra over a scheme in [Rue23]. I will use this new definition to study the theory of triality over a scheme, which I detail in the following sections.

1.B. **Triality without Octonions.** Triality is usually described using symmetric composition algebras of dimension 8, which are closely related to octonion algebras. An octonion algebra comes with an involution, denoted  $x \mapsto \overline{x}$ , and modifying the multiplication into a new *para-multiplication* by  $x \star y = \overline{x} \cdot \overline{y}$  produces a symmetric composition algebra. These are non-associative algebras. Given a symmetric composition algebra  $(S, \star, n)$  over a field  $\mathbb{F}$  of characteristic not 2, which comes with a multiplicative quadratic form n, there is a key isomorphism of [KMRT, 35.1]

$$\operatorname{Cl}_0(S,n) \xrightarrow{\sim} \operatorname{End}_{\mathbb{F}}(S) \times \operatorname{End}_{\mathbb{F}}(S)$$

from the even Clifford algebra of (S, n) to two copies of the endomorphism algebra. This isomorphism is defined using left and right multiplication operators with respect to the multiplication  $\star$ . The phenomena of triality can then be deduced from this isomorphism using properties of the symmetric composition multiplication.

In [DQ, 4.5], they show that the isomorphism above also aligns the canonical quadratic pair on the Clifford algebra with the quadratic pairs adjoint to n on the right, therefore producing an isomorphism

$$(\operatorname{Cl}_0(S,n), \underline{\sigma_{n_0}}, f_n) \xrightarrow{\sim} (\operatorname{End}_{\mathbb{F}}(S), \sigma_n, f_n) \times (\operatorname{End}_{\mathbb{F}}(S), \sigma_n, f_n)$$

of algebras with quadratic pair. From this, they show that triality can also be described in characteristic 2. Since I now have a definition of the canonical quadratic pair over a scheme, it should be routine to check that all of the above also holds in this more general setting, thus providing a description of triality over schemes.

However, I am also interested in the following, potentially alternative, approach. Let V be an  $\mathbb{F}$ -vector space and denote by  $\mathbb{H}(V) = V \oplus V^*$  the direct sum of V and its dual. This has a canonical hyperbolic quadratic form defined by q(x, f) = f(x). By [Knus, IV.2.1.1], there is an isomorphism

$$\operatorname{Cl}(\mathbb{H}(V), q) \xrightarrow{\sim} \operatorname{End}_{\mathbb{F}}(\wedge V)$$

from the Clifford algebra to the vector space endomorphisms of the wedge algebra of V, defined using the wedge multiplication in  $\wedge V$ . By [CF, 4.2.0.10], an analogue of this isomorphism exists over schemes as well. This isomorphism restricts to an isomorphism with the even Clifford algebra,

$$\operatorname{Cl}_0(\mathbb{H}(V), q) \xrightarrow{\sim} \operatorname{End}_{\mathbb{F}}(\wedge_0 V) \times \operatorname{End}_{\mathbb{F}}(\wedge_1 V),$$

where  $\wedge_0 V$  denotes the vector space spanned by wedge products of even length and  $\wedge_1 V$  those of odd length. When V is 4-dimensional, so  $\mathbb{H}(V)$  is 8-dimensional, the vectors spaces  $\mathbb{H}(V)$ ,  $\wedge_0 V$ , and  $\wedge_1 V$  are noncanonically isomorphic to one another and so we get a situation something like the one above. While there is no symmetric composition algebra structure to exploit here, I believe that using Chevalley generators and techniques from [SGA3], one should be able to identify triality within this isomorphism as well.

Given a split linear algebraic group G, it has a root system  $\Phi$ , and *Chevalley generators* are a choice of elements  $x_{\alpha}(t) \in G$ , indexed by roots  $\alpha \in \Phi$  and scalars  $t \in \mathbb{F}$ , subject to relations determined by the geometry of the root system. For example, consider the special orthogonal group

$$\mathbf{SO}_{2n} = \{B \in \mathcal{M}_{2n}(\mathbb{F}) \mid \Omega B^T \Omega = B^{-1}\} \text{ where } \Omega = \begin{bmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{bmatrix}$$

It is of type  $D_n$ , which has root system  $\Phi = \{\pm e_i \pm e_j \mid 1 \le i, j \le n, i < j\}$ . One choice of Chevalley generators, given in [Rue21, 1.3.21], is

$$\begin{aligned} x_{e_i - e_j}(t) &= I + t(E_{ij} - E_{j\bar{i}}) \\ x_{e_i + e_j}(t) &= I + t(E_{i\bar{j}} - E_{j\bar{i}}) \\ x_{e_i + e_j}(t) &= I + t(E_{i\bar{j}} - E_{j\bar{i}}) \\ \end{aligned}$$

where  $\overline{i} = 2n + 1 - i$  and  $E_{ij}$  is the matrix with 1 in the (i, j) entry and zeroes elsewhere. By [SGA3, XXIV.1.3(iii)], outer automorphisms of the group arise from graph automorphisms of the Dynkin diagram. A graph automorphism corresponds to an automorphism  $\varphi \colon \Delta \xrightarrow{\sim} \Delta$  of some simply system for the root system  $\Phi$ , and by defining  $x_{\alpha}(t) \mapsto x_{\varphi(\alpha)}(t)$  and  $x_{-\alpha}(t) \mapsto x_{-\varphi(\alpha)}(t)$  for  $\alpha \in \Delta$ , this extends uniquely to an automorphism of G. Therefore, I think that by explicitly computing the images of Chevalley generators for the spin group, which sits inside the Clifford algebra, under the isomorphism above, one should be able to identify that an order three outer automorphism is being induced. If this is the case, then all following phenomena of triality can be explained without appealing to octonions or symmetric composition algebras.

In any case, once triality is obtained over a scheme by either method, I then plan to develop the resulting theory of trialitarian triples and trialitarian algebras over a scheme. This would provide an analogue to the results of [KMRT, Ch. X], including a classification of  $\mathbf{PGO}_8^+$ -torsors over a scheme in terms of triples as well as a classification of  $\mathbf{PGO}_8^+ \rtimes \mathbb{S}_3$ -torsors in terms of trialitarian algebras, which would also classify all groups of type  $D_4$ .

1.C. Half-spin groups. The spin group of a hyperbolic quadratic form sits inside the even Clifford algebra,  $\operatorname{Spin}_q \subset \operatorname{Cl}_0(\mathbb{H}(V), q)$ . Under the isomorphism

$$\operatorname{Cl}_0(\mathbb{H}(V), q) \xrightarrow{\sim} \operatorname{End}_{\mathbb{F}}(\wedge_0 V) \times \operatorname{End}_{\mathbb{F}}(\wedge_1 V)$$

appearing above, the projection of  $\mathbf{Spin}_q$  onto either of the two factors produces the half-spin group. When dealing with these groups as sheaves, this means half-spin is defined as an image sheaf involving sheafification and therefore it is not clear how it appears over a given coefficient ring or field. While this research goal is more nebulous, I would like to use any insights gained from the computations used to study triality as proposed in the previous section in order to better understand half-spin groups of any rank. Perhaps this could lead to a more satisfying description of the half-spin groups than currently exists. Additionally, since these calculations would occur over a scheme, then may lend themselves to generalizing results from [Rue20] relating to cohomological invariants of the half-spin groups.

## 2. Stacks and Cohomology

In another forthcoming project with Erhard Neher and Philippe Gille, we are building off our work on quadratic pairs in order to generalize the equivalence " $A_1 \times A_1 \equiv D_2$ ." This equivalence occurs in many forms. First, it occurs as an equivalence of root systems, both  $A_1 \times A_1$  and  $D_2$  have Dynkin diagrams which are two disjoint points. For a field  $\mathbb{F}$ , it also appears in [KMRT, 15.B] as an equivalence of categories between the groupoid of degree 2 Azumaya algebras over a degree 2 étale extension of  $\mathbb{F}$  (type  $A_1 \times A_1$ ) and the groupoid of degree 4 Azumaya algebras over  $\mathbb{F}$  with a quadratic pair (type  $D_2$ ). This also implies an equivalence between the associated groupoids of linear algebraic groups of types  $A_1 \times A_1$  and type  $D_2$ . The equivalence is demonstrated in [KMRT] using a norm functor, which is a generalization of the corestriction functor with respect to a finite separable field extension coming from Galois cohomology. If  $\mathbb{K}/\mathbb{F}$  is such a field extension, there is an inclusion of Galois groups  $Gal(\mathbb{F}_{sep}, \mathbb{K}) \subseteq Gal(\mathbb{F}_{sep}, \mathbb{F})$  and thus a corestriction map

$$H^r(\operatorname{Gal}(\mathbb{F}_{\operatorname{sep}},\mathbb{K}),\mathbb{F}_{\operatorname{sep}}^{\times}) \to H^r(\operatorname{Gal}(\mathbb{F}_{\operatorname{sep}},\mathbb{F}),\mathbb{F}_{\operatorname{sep}}^{\times}).$$

When r = 2, this becomes a corestriction map between Brauer groups  $Br(\mathbb{K}) \to Br(\mathbb{F})$ . In [Rie], Riehm constructs a corestriction functor from the category of central simple  $\mathbb{K}$ -algebras of degree r to the category of central simple  $\mathbb{F}$ -algebras of degree  $r^{[\mathbb{K}:\mathbb{F}]}$  that recovers the map on Brauer groups after passing to Brauer equivalence. This is an example of a cohomological fact later being described as induced by a functor on underlying objects. The norm functor appearing in [KMRT] is essentially an application of the norm functor of Knus and Ojanguren from [KO], which works for finite étale ring extensions. Both the functors in [Rie] and in [KO] were generalized by Ferrand in [Fer] to a norm functor with respect to finite locally free ring extensions.

In our project, we are further generalizing the norm functor to the setting of finite locally free extensions of schemes  $T \to S$ . Initially, we produce a functor  $\mathcal{N}_{T/S} : \mathfrak{QCoh}_{\mathcal{O}_T} \to \mathfrak{QCoh}_{\mathcal{O}_S}$  from the category of quasicoherent modules (on the big fppf site) over T to the category of quasi-coherent modules over S. It is constructed from, and therefore generalizes, the functor in [Fer]. Second, we show that as the finite locally free extension varies, these functors form a morphism of stacks  $\mathcal{N} : \mathfrak{QCoh}_{\mathrm{flf}} \to \mathfrak{QCoh}$ . Here, omitting mention of the morphisms,  $\mathfrak{QCoh}$  denotes the stack of quasi-coherent sheaves over  $\mathfrak{Sch}_S$ , i.e., it has objects of the form  $(T, \mathcal{M})$  where  $T \in \mathfrak{Sch}_S$  and  $\mathcal{M} \in \mathfrak{QCoh}_{\mathcal{O}_T}$ . The first stack,  $\mathfrak{QCoh}_{\mathrm{flf}}$ , is comprised of objects of the form

$$(T' \to T, \mathcal{M}')$$

where  $T' \to T$  is a finite locally free morphism in  $\mathfrak{Sch}_S$  and  $\mathcal{M}' \in \mathfrak{QCoh}_{\mathcal{O}_{T'}}$ . The stack morphism then behaves on objects as  $\mathcal{N}(T' \to T, \mathcal{M}') = (T, \mathcal{N}_{T'/T}(\mathcal{M}'))$ . The advantage of generalizing to the level of stacks is that when we restrict the norm functor to subgerbes, as we will do when generalizing  $A_1 \times A_1 \equiv D_2$ , we may take advantage of the following lemma from [Gir].

**Lemma 2.1** ([Gir, III.2.5.3]). Let  $\varphi : \mathfrak{F} \to \mathfrak{G}$  be a morphism of gerbes. For  $x \in \mathfrak{F}(S)$  there is an associated morphism of group sheaves  $\varphi_x : \operatorname{Aut}_{\mathfrak{F}}(x) \to \operatorname{Aut}_{\mathfrak{G}}(\varphi(x))$ . Then, the map on first cohomology induced by  $\varphi_x$  is the map

$$H^{1}(S, \operatorname{Aut}_{\mathfrak{F}}(x)) \to H^{1}(S, \operatorname{Aut}_{\mathfrak{G}}(\varphi(x)))$$
$$[y] \mapsto [\varphi(y)]$$

where we identify  $H^1(S, \operatorname{Aut}_{\mathfrak{F}}(x))$  with the set of isomorphism classes in  $\mathfrak{F}(S)$ , and  $H^1(S, \operatorname{Aut}_{\mathfrak{G}}(\varphi(x)))$  with the isomorphism classes in  $\mathfrak{G}(S)$ .

This allows us to extract various cohomological statements arising from the norm functor, in particular, linking it to the *Segre embedding* 

$$(\mathbf{GL}_n)^d \to \mathbf{GL}_{n^d}$$
  
 $(B_1, \dots, B_n) \mapsto B_1 \otimes \dots \otimes B_d$ 

where the tensor product denotes the tensor as linear maps. Conversely, by restricting the Segre homomorphism to an isomorphism between appropriate groups, we will show that there is an equivalence  $A_1 \times A_1 \equiv D_2$  of stacks between the gerbe of degree 2 Azumaya algebras over a degree 2 étale extension of the base scheme and the gerbe of degree 4 Azumaya algebras over S with a quadratic pair.

Beyond this current project, I am interested in continuing to use known cohomological facts as indication of stack morphisms to be found and conversely using new stack morphisms to produce cohomological statements. In particular, after triality is established over schemes, I expect to obtain analogues of the cohomological statements given after [KMRT, 42.1] and in [KMRT, 44.5]. Philosophically, there should then be a gerbe of trialitarian triples and a gerbe of trialitarian algebras along with stack morphisms which induce the cohomological statements. As part of my research on triality over schemes, I will work on finding these stack theoretic descriptions.

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