Research Statement
Cameron Ruether, Ph.D.

Postdoctoral Fellow
Email: cameronruether@gmail.com
Website: cameron-ruether.bitbucket.io

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, NL, Canada
A1C 5S7

My research is focused on linear algebraic groups and their cohomological invariants. In recent work with Philippe Gille and Erhard Neher [GNR], we study quadratic pairs on Azumaya algebras over an arbitrary base scheme. Below I give a summary, more details can be found in section 1. Given an Azumaya algebra $\mathcal{A}$, a quadratic pair (σ, f) is an orthogonal involution σ and a linear function f in the dual of the symmetric elements of $\mathcal{A}$. The function f encodes information about how σ arises from quadratic forms, which does not always happen if 2 is not invertible over the base scheme. Because of this, quadratic pairs are needed to define semisimple group schemes of type D, such as the special orthogonal group, in settings where 2 may not be invertible. In our work we define two cohomological obstructions associated to an Azumaya algebra with orthogonal involution $(\mathcal{A}, \sigma)$ called the strong obstruction and the weak obstruction. We show that the weak obstruction is trivial if and and only if $(\mathcal{A}, \sigma)$ can be equipped with a quadratic pair, and we show that the strong obstruction is trivial if and only if $(\mathcal{A}, \sigma)$ can be equipped with a special type of quadratic pair which is defined by a global section of $\mathcal{A}$. This means that the weak obstruction is trivial whenever the strong obstruction is trivial.

In my Ph.D. thesis, which works over a field of characteristic not 2, I showed how twisting and Galois descent techniques can be applied to the split spin and half-spin group schemes, **Spin** and **HSpin**, to produce their non-split counterparts. In doing so, I developed an analogous theory of twisting for Hopf algebras. These twisting techniques were then used to construct new homomorphisms between split and non-split linear algebraic groups which mirror the tensor product map $A_1 \times A_2 \to A_1 \otimes A_2$ between algebras. I used these new maps in the split case to compute the group of degree three cohomological invariants of $\mathbf{HSpin}_{4n}$ for all $n > 1$ in [Rue]. My future research goals include using my work on quadratic pairs to study groups of type D and their cohomological invariants in settings other than characteristic not 2. I will investigate whether similar techniques to the ones I used in my thesis can be used, or if more complicated behaviour arises over general schemes.

1. Quadratic Pairs

Quadratic pairs on central simple algebras over a field $\mathbb{F}$ of any characteristic were introduced in [KMRT, §5]. They were motivated by the following fact about quadratic forms and bilinear forms. Let V be a non-zero finite dimensional $\mathbb{F}$–vector space. Given a quadratic form $q\colon V \to \mathbb{F}$, its polar form $b_q(x, y) = q(x + y) - q(x) - q(y)$ is a symmetric bilinear form on V. Conversely, given a bilinear form $b\colon V \times V \to \mathbb{F}$, the function $q_b(x) = b(x, x)$ is a quadratic form on V. When $\mathbb{F}$ is not of characteristic 2, the above constructions provide a bijection between symmetric bilinear forms and quadratic forms. However, if $\mathbb{F}$ is of characteristic 2, this correspondence breaks down. In particular, there exist symmetric bilinear forms which are not the polar of any quadratic form. Quadratic pairs illuminate this behaviour for regular bilinear forms, i.e. those bilinear forms such that the map from V to its dual $b^*\colon V \to V^*$ given by $x \mapsto b(x, \_\_)$ is an isomorphism. Regular bilinear forms have an associated adjoint anti-automorphism σ on $\mathrm{End}_{\mathbb{F}}(V)$ uniquely defined by

$$b(x, By) = b(\sigma(B)x, y)$$

for all $x, y \in V$ and $B \in \mathrm{End}_{\mathbb{F}}(V)$. This adjoint anti-automorphism is an involution, that is $\sigma^2 = \mathrm{Id}$, whenever b is symmetric or skew-symmetric. In those cases we call σ *orthogonal* if b is symmetric, and we call it *weakly-symplectic* if b is skew-symmetric. Additionally, we call σ *symplectic* if b is alternating, that is $b(x, x) = 0$ for all $x \in V$. Conversely, any involution on $\mathrm{End}_{\mathbb{F}}(V)$ is the adjoint involution of some symmetric or skew-symmetric regular bilinear form. We discuss below how this correspondence allows us to view quadratic pairs as attaching extra information to a bilinear form. The definition of a quadratic pair is the following.

Definition 1. *Let A be a central simple algebra over $\mathbb{F}$. A* quadratic pair *on A is a pair (σ, f) where σ is an orthogonal involution on A, and $f\colon \mathrm{Sym}(A,\sigma) \to \mathbb{F}$ is a linear function from the symmetric elements of A (those $a \in A$ satisfying $\sigma(a) = a$) such that*

$$f(a + \sigma(a)) = \mathrm{Trd}_A(a)$$

for all $a \in A$.

In the above definition, Trd is the reduced trace of A, which is the usual trace when $A = \mathrm{End}_{\mathbb{F}}(V)$.

Note that quadratic pairs are only interesting when 2 is not invertible. For any symmetric element $s \in \mathrm{Sym}(A,\sigma)$ we have $\mathrm{Trd}_A(s) = f(s+\sigma(s)) = 2f(s)$ and so if 2 is invertible we must have $f = \frac{1}{2}\mathrm{Trd}_A$. Conversely given any orthogonal involution, setting $f = \frac{1}{2}\mathrm{Trd}_A$ will extend it to a quadratic pair. This is the reflection of the fact that we have a correspondence between symmetric bilinear forms and quadratic forms in characteristic not 2.

Consider an orthogonal involution σ on the central simple algebra $\mathrm{End}_{\mathbb{F}}(V)$, and let b be a symmetric bilinear form on V whose adjoint involution is σ. By [KMRT, 5.10], the bilinear form gives rise to an isomorphism

$$\begin{aligned} \varphi_b\colon V \otimes_{\mathbb{F}} V &\xrightarrow{\sim} \mathrm{End}_{\mathbb{F}}(V) \\ x \otimes y &\mapsto (v \mapsto b(x,v)y). \end{aligned}$$

and then by [KMRT, 5.11], if (σ, f) is a quadratic pair on $\mathrm{End}_{\mathbb{F}}(V)$, then $q(v) = f(\varphi_b(v \otimes v))$ is a quadratic form whose polar is b. In this way quadratic pairs indicate when b is the polar of a quadratic.

In [KMRT, §23.B], the authors go on to use quadratic pairs on general central simple algebras with orthogonal involution to define semisimple linear algebraic groups of type D in any characteristic, in particular in characteristic 2, and quadratic pairs are a necessary part of this construction.

The study of quadratic pairs was generalized by Calmés and Fasel in [CF] from the setting of central simple algebras over a field, to the setting of Azumaya algebras over an arbitrary base scheme. In particular, working over the scheme S, they work with sheaves on the ringed site $(\mathfrak{Sch}_S, \mathcal{O})$ where $\mathfrak{Sch}_S$ is the big fppf-site of schemes over S, and $\mathcal{O}$ is the sheaf

$$\begin{aligned} \mathcal{O}\colon \mathfrak{Sch}_S &\to \mathfrak{Rings} \\ T &\mapsto \mathcal{O}_T(T) \end{aligned}$$

which returns the global sections of a scheme's structure sheaf. Vector spaces are replaced with locally free $\mathcal{O}$–modules of finite rank, and central simple algebras are replaced with Azumaya algebras, which are $\mathcal{O}$–algebras such that for every affine variety $U \in \mathfrak{Sch}_S$, the algebra $\mathcal{A}(U)$ is an Azumaya algebra over the ring $\mathcal{O}(U)$ as in [Ford]. For an $\mathcal{O}$–module $\mathcal{M}$ we denote its sheaf of internal endomorphisms by $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$. An involution is a natural transformation $\sigma\colon \mathcal{A} \to \mathcal{A}$ which is an $\mathcal{O}$–linear anti-automorphism of order 2 (for $\mathcal{A} = \mathcal{O}$ we allow $\sigma = \mathrm{Id}$). Locally with respect to the fppf topology on $\mathfrak{Sch}_S$, an Azumaya algebra $\mathcal{A}$ will be isomorphic to $\mathcal{E}nd_{\mathcal{O}}(\mathcal{M})$ for a locally free $\mathcal{O}$–module $\mathcal{M}$ of finite positive rank. If $\mathcal{A}$ has an involution, then locally it will be the adjoint involution of some regular bilinear form on $\mathcal{M}$. Once again, we call σ *orthogonal* if this bilinear form is symmetric, *weakly-symplectic* if this bilinear form is skew-symmetric, and *symplectic* if this bilinear form is alternating. The notion of a quadratic pair is essentially the same in this setting.

Definition 2. *Let $\mathcal{A}$ be an Azumaya algebra over S. A* quadratic pair *is a pair (σ, f) where σ is an orthogonal involution on $\mathcal{A}$, and $f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ is an $\mathcal{O}$–linear map from the sheaf of symmetric elements such that*

$$f(a + \sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(a)$$

for all $T \in \mathfrak{Sch}_S$ and all sections $a \in \mathcal{A}(T)$.

In my recent work with Philippe Gille and Erhard Neher [GNR], we studied how local properties of quadratic pairs fail to globalize, and we introduced cohomological obstructions which measure this failure. In particular, we gave a cohomological description of which orthogonal involutions can be extended to a quadratic pair.

To describe these cohomological obstructions, we first fix some notation for the following submodules of an Azumaya algebra with involution $(\mathcal{A},\sigma)$.

$$\begin{aligned} \mathcal{S}kew_{\mathcal{A},\sigma} &= \mathrm{Ker}(\mathrm{Id}+\sigma) \quad (\textit{skew-symmetric elements}),\\ \mathcal{A}lt_{\mathcal{A},\sigma} &= \mathrm{Img}(\mathrm{Id}-\sigma) \quad (\textit{alternating elements}),\\ \mathcal{S}ymd_{\mathcal{A},\sigma} &= \mathrm{Img}(\mathrm{Id}+\sigma) \quad (\textit{symmetrized elements}), \end{aligned}$$

where $\mathrm{Id}+\sigma$ and $\mathrm{Id}-\sigma$ are morphisms $\mathcal{A} \to \mathcal{A}$, and $\mathrm{Img}(\_\_)$ is the image fppf-sheaf. We show in [GNR, 6.1] that having $1_\mathcal{A} \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$ is a necessary condition for $(\mathcal{A},\sigma)$ to be equipped with a quadratic pair. We call those σ with this property *locally quadratic* involutions, and our new cohomological obstructions are defined for these involutions. For any Azumaya algebra with orthogonal involution there is an exact sequence of sheaves

$$0 \to \mathcal{S}kew_{\mathcal{A},\sigma} \to \mathcal{A} \to \mathcal{S}ymd_{\mathcal{A},\sigma} \to 0$$

and therefore a long exact cohomology sequence

$$0 \to \mathcal{S}kew_{\mathcal{A},\sigma}(S) \to \mathcal{A}(S) \to \mathcal{S}ymd_{\mathcal{A},\sigma}(S) \xrightarrow{\delta} H^1_{\mathrm{fppf}}(S,\mathcal{S}kew_{\mathcal{A},\sigma}) \to \dots$$

with connecting map δ. Further, since $\mathcal{A}lt_{\mathcal{A},\sigma} \subseteq \mathcal{S}kew_{\mathcal{A},\sigma}$, there is a natural projection map $\mathcal{S}kew_{\mathcal{A},\sigma} \to \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma}$ and therefore an induced map

$$\phi\colon H^1_{\mathrm{fppf}}(S,\mathcal{S}kew_{\mathcal{A},\sigma}) \to H^1_{\mathrm{fppf}}(S,\mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma}).$$

When $(\mathcal{A},\sigma)$ is locally quadratic we may discuss $\delta(1_\mathcal{A})$ and $\phi(\delta(1_\mathcal{A}))$.

Definition 3. *Let $(\mathcal{A},\sigma)$ be an Azumaya algebra with locally quadratic involution. Then we call*

(1) $\Omega(\mathcal{A},\sigma) = \delta(1_\mathcal{A}) \in H^1_{\mathrm{fppf}}(S,\mathcal{S}kew_{\mathcal{A},\sigma})$ the strong obstruction*, and*
(2) $\omega(\mathcal{A},\sigma) = \phi(\Omega(\mathcal{A},\sigma)) \in H^1_{\mathrm{fppf}}(S,\mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}lt_{\mathcal{A},\sigma})$ the weak obstruction*.*

When these cohomological obstructions are non-trivial, they prevent the existence of quadratic pairs in the following way.

Theorem 1 ([GNR, 6.8])**.** *Let $(\mathcal{A},\sigma)$ be an Azumaya algebra with a locally quadratic involution.*

(1) There exists a linear map $f\colon \mathcal{A} \to \mathcal{O}$ such that $(\mathcal{A},\sigma,f|_{\mathcal{S}ym_{\mathcal{A},\sigma}})$ is an Azumaya algebra with quadratic pair if and only if $\Omega(\mathcal{A},\sigma) = 0$. In this case $f = \mathrm{Trd}_\mathcal{A}(\ell\_\_)$ for an element $\ell \in \mathcal{A}(S)$ with $\ell + \sigma(\ell) = 1$.
(2) There exists a linear map $f\colon \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ such that $(\mathcal{A},\sigma,f)$ is an Azumaya algebra with quadratic pair if and only if $\omega(\mathcal{A},\sigma) = 0$.

Furthermore, we show that these obstructions can be non-trivial by constructing an example where $\Omega(\mathcal{A},\sigma) \neq 0$ and $\omega(\mathcal{A},\sigma) = 0$ [GNR, 7.1], and an example where $\omega(\mathcal{A},\sigma) \neq 0$, and therefore also $\Omega(\mathcal{A},\sigma) \neq 0$, [GNR, 7.4].

2. Linear Algebraic Groups and Twisting

In my Ph.D. thesis, while working over a field $\mathbb{F}$ of characteristic not 2, I studied linear algebraic groups schemes, also simply called linear algebraic groups, which are generalizations of subgroups of matrix algebras. We view linear algebraic groups as in [KMRT, §20.A]. A linear algebraic group $\mathbf{G}$ is a functor $\mathbf{G}\colon \mathbf{Alg}_\mathbb{F} \to \mathbf{Grp}$ from unital, commutative, associative $\mathbb{F}$-algebras to groups. The functor must be representable, and it will be represented by a Hopf algebra. For example, the split special linear group scheme is the functor

$$\begin{aligned}\mathbf{SL}_n\colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp}\\ R &\mapsto \{B \in \mathrm{M}_n(R) \mid \det(B) = 1\},\end{aligned}$$

and if A is a non-split, i.e., non-matrix central simple algebra, then the non-split special linear group scheme is the functor

$$\begin{aligned}\mathbf{SL}(A)\colon \mathbf{Alg}_{\mathbb{F}} &\to \mathbf{Grp}\\ R &\mapsto \{x \in A \otimes_{\mathbb{F}} R \mid \mathrm{Nrd}(x) = 1\}\end{aligned}$$

where Nrd is the reduced norm map of A. The split version, so called because it contains a split maximal torus $T \cong \mathbb{G}_{\mathrm{mult}}^n$ where $\mathbb{G}_{\mathrm{mult}} : R \mapsto R^\times$ is the multiplicative group, is the version which come from generalizing the familiar subgroup of $\mathrm{M}_n(\mathbb{F})$. Likewise, there are other generalizations of familiar groups both in split and non-split versions, for example the symplectic groups **Sp** and **PSp** are associated to central simple algebras with symplectic invoulution. Of particular interest to us are the groups of type D, which are **Spin**, **HSpin**, **SO**, and **PSO** associated to even rank central simple algebras with orthogonal involution.

It is well known that central simple algebras are twisted forms of matrix algebras. For example, since the automorphism group of $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ is the projective linear group, $\mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$, central simple algebras of degree n are classified by the first Galois cohomology set $H^1(\mathbb{F}, \mathbf{PGL}_n)$. An element $[\alpha] \in H^1(\mathbb{F}, \mathbf{PGL}_n)$ is represented by a cocycle $\alpha\colon \mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F}) \to \mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$, and it corresponds to a central simple algebra in the following way. The Galois group acts naturally on $\mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$ by acting on entries, and this action can be twisted by α to define a new action. For $\sigma \in \mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})$ and $B \in \mathrm{M}_n(\mathbb{F}_{\mathrm{sep}})$, the twisted action is

$$\sigma \cdot_\alpha B = \alpha_\sigma(\sigma(B)).$$

The fixed points of this new action are then a central simple algebra over $\mathbb{F}$. Similarly, central simple algebras with orthogonal involution are classified by $H^1(\mathbb{F}, \mathbf{PSO}_n)$, and those with symplectic involution by $H^1(\mathbb{F}, \mathbf{PSp}_{2n})$. It is also known that twisting and descent in this manner can be used to produce non-split linear algebraic groups from their split counterparts. For example, if $[\alpha] \in H^1(\mathbb{F}, \mathbf{PGL}_n)$, we can twist the natural Galois action on $\mathbf{PGL}_n(\mathbb{F}_{\mathrm{sep}})$ via

$$\sigma \cdot_\alpha [B] = \alpha_\sigma [B] (\alpha_\sigma)^{-1}$$

and then the fixed points are $\mathbf{PGL}(A)(\mathbb{F})$. This descent from $\mathbb{F}_{\mathrm{sep}}$ to $\mathbb{F}$ can be repeated for any input $R \in \mathbf{Alg}_{\mathbb{F}}$ by using the image of α in $\mathbf{PGL}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ to twist the action on $\mathbf{PGL}_n(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$, the fixed points of which will be $\mathbf{PGL}(A)(R)$. Similar twisting can also produce non-split **SO**, **PSO**, **Sp**, and **PSp** from their split counterparts. These examples follow more or less immediately from the story of descent for central simple algebras since these groups are either subgroups of, or automorphism groups of, central simple algebras.

I showed in my thesis that this process also works to produce non-split **Spin** and **HSpin** groups from their split counterparts. To do so for **HSpin**, I needed to exploit the functorial correspondence between group schemes and their representing Hopf algebras. In particular, I gave a concrete description of how the twisting occurring within the linear algebraic groups is mirrored functorially in their Hopf algebras. In particular, let $\mathbf{G}$ be a linear algebraic group with a twisted form $\mathbf{G}(A)$ related to an algebra A. If A corresponds to a cocycle α, then after twisting the Galois action on $\mathbf{G}(R \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ via conjugation by α, the fixed points of the action are $\mathbf{G}(A)(R)$. The group $\mathbf{G}$ has a Hopf algebra H. I described how one can compute an analogous cocycle $\mathfrak{a}\colon \mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F}) \to \mathrm{Aut}(H \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})$ from α such that, after using $\mathfrak{a}$ to twist the action, we get fixed points

$$(H \otimes_{\mathbb{F}} \mathbb{F}_{\mathrm{sep}})^{\mathrm{Gal}(\mathbb{F}_{\mathrm{sep}}/\mathbb{F})_{\mathfrak{a}}} = H(A)$$

where $H(A)$ is the Hopf algebra of the twisted form $G(A)$.

3. Cohomological Invariants

Cohomological invariants of linear algebraic groups have been the topic of ongoing study since they were introduced by Serre. These invariants can be defined as in [GMS], which we now review. Let G be a linear algebraic group defined over a field $\mathbb{F}$. We consider two functors from the category of field extensions over $\mathbb{F}$. First is the Galois cohomology functor $H^1(-,G)$, taking a field $\mathbb{E}/\mathbb{F}$ to $H^1(\mathbb{E},G)$, and second is any functor $A\colon \mathbf{Fields}_{\mathbb{F}} \to \mathbf{Abelian\ Groups}$. A *cohomological invariant of G with coefficients in A* is then a natural transformation between these functors $H^1(-,G) \to A$. Of particular interest has been the study of cohomological invariants with values in $H^{d+1}(-,\mathbb{Q}/\mathbb{Z}(d))$, called *cohomological invariants of degree* $d+1$. Here, $\mathbb{Q}/\mathbb{Z}(d)$ is in essence the tensor product of d copies of the group of all roots of unity of $\mathbb{F}_{\text{sep}}$, with a modified component replacing p^{th} roots when $\mathbb{F}_{\text{sep}}$ is characteristic p. It is fully defined in [GMS, Appendix A]. Cohomological invariants of degree $d+1$ form a group denoted by $\text{Inv}^{d+1}(G,d)$.

Invariants of degree two are well understood, they correspond to central characters of the group, so recent focus has been on invariants of degree three, such as in the work of Baek, Bermudez and Ruozzi, Merkurjev, Neshitov, and Zainoulline. There is a subgroup of these degree three invariants, called the *normalized invariants* and denoted by $\text{Inv}^3(G,2)_{\text{norm}}$, which are those invariants sending the trivial object, the constant cocyle $\alpha(\sigma) = 1$, of $H^1(-,G)$ to zero in $H^3(-,\mathbb{Q}/\mathbb{Z}(2))$. In the case where G is absolutely simple and simply connected, such as when $G = \mathbf{SL}_n$ or $G = \mathbf{Spin}_n$, this group is finite and cyclic, generated by the *Rost invariant* [GMS, pg. 129]. Outside of this case, Merkurjev described the normalized invariants for semisimple adjoint groups of inner type in [Mer], such as $\mathbf{PGL}(A)$ for a central simple algebra A over $\mathbb{F}$. In [BR], Bermudez and Ruozzi computed some related groups of invariants, called decomposable and indecomposable invariants, for some split semisimple groups which are neither simply connected nor adjoint. In particular they computed those invariants for the groups $\mathbf{HSpin}_{4n}$. Using their computations and a map from $\mathbf{HSpin}_{16}$ to a group of type E_8, they were able to describe the structure of the normalized invariants of $\mathbf{HSpin}_{16}$. However, because of the reliance on the group of type E_8, which is the largest exceptional type, their techniques could not be extended beyond the case of $n = 4$.

In [Rue], working over a field of characteristic not 2, I generalized the results of Bermudez and Ruozzi and I described the structure of the normalized invariants of $\mathbf{HSpin}_{4n}$ for all $n > 1$.

$$\text{Inv}^3(\mathbf{HSpin}_{4n},2)_{\text{norm}} \cong \begin{cases} \mathbb{F}/(\mathbb{F}^\times)^2 & n > 1 \text{ is odd or } n = 2 \\ \mathbb{F}/(\mathbb{F}^\times)^2 \oplus \mathbb{Z}/2\mathbb{Z} & n \equiv 2 \pmod 4 \text{ and } n \neq 2 \\ \mathbb{F}/(\mathbb{F}^\times)^2 \oplus \mathbb{Z}/4\mathbb{Z} & n \equiv 0 \pmod 4. \end{cases}$$

To do so, I generalized a technique of Garibaldi [Gar], who constructed a tensor product style map $\mathbf{PSp}_2 \times \mathbf{PSp}_8 \hookrightarrow \mathbf{HSpin}_{16}$, to construct maps $\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m} \hookrightarrow \mathbf{HSpin}_{4nm}$ for all n and m. Then by using the induced map

$$\text{Inv}^3(\mathbf{HSpin}_{4nm},2)_{\text{norm}} \to \text{Inv}^3(\mathbf{PSp}_{2n} \times \mathbf{PSp}_{2m},2)_{\text{norm}}$$

together with some simple group theoretic arguments, I obtained the result above. The assumption that the characteristic of the field is not 2 was likely unnecessary, and I suspect that these results remain the same in characteristic 2.

4. Research Goals

(1) My main research interests involve continuing my study of quadratic pairs, their associated group schemes, and cohomological invariants. In particular, I am interested in understanding these groups when they are defined in settings where 2 may not be invertible, such as over a general base scheme. For example, an interesting phenomenon which occurs when $2 = 0$ over the base scheme is that the notion of orthogonal and weakly-symplectic involutions coincide. By [GNR, 5.5], if $(\mathcal{A},\sigma)$ is an Azumaya algebra with symplectic involution, it is also an algebra with locally quadratic

orthogonal involution. If this algebra can be completed with a quadratic pair $(\mathcal{A}, \sigma, f)$, then the orthogonal group of the quadratic pair sits as a subgroup inside the symplectic group,

$$\mathbf{O}(\mathcal{A}, \sigma, f) \subseteq \mathbf{Sp}(\mathcal{A}, \sigma).$$

However, not all such symplectic involutions can be extended to a quadratic pair, so one may ask: Under which conditions on the symplectic group can the involution σ be extended to a quadratic pair, i.e., how can the weak obstruction $\omega(\mathcal{A}, \sigma)$ be characterized in terms of the symplectic group? Additionally, there are many other research topics my co-authors and I intend to explore as a continuation of the work in [GNR]. For example, we expect that the classical exceptional isomorphism $A_1 \times A_1 \cong D_2$ also holds over schemes, and gives an equivalence of categories between the category of quaternion algebras over a quadratic étale extension of the base scheme and the category of Azumaya algebras of degree 4 with quadratic pair. We intend to work out the details of this correspondence. More generally, we suspect that continuing our study will lead to a notion of Morita theory for quadratic pairs.

(2) I am also interested more broadly in studying generalizations of cohomological invariants to the setting over a base scheme. Similar ideas have already appeared, such as in [EKLV], which discusses invariants taking values in the étale cohomology of base schemes. The classical notion of a cohomological invariant uses Galois cohomology and therefore concerns itself with functors from the category of field extensions of the base field. In the more general context of working with groups which are sheaves on the fppf site $\mathfrak{Sch}_S$, one could ask about the structure of natural transformations from the first cohomology functor $H^1_{\text{fppf}}(\_\_, G)$. That is, for another functor $H \colon \mathfrak{Sch}_S \to \mathfrak{Ab}$, one can ask about the group $\mathrm{Inv}_{\text{fppf}}(G, H)$, perhaps called the group of *flat cohomological invariants*. So far in my research, many of the classical results which were stated just for groups over fields have natural generalizations which hold in the setting over schemes, and so I believe this would both an interesting and fruitful direction of research.

(3) I also remain interested in groups of type D over a field. Since the degree three normalized invariants are known for simply connected groups and adjoint groups of inner type, and groups which are neither simply connected nor adjoint only occur in the classical types A and D, there are only a few groups remaining where these invariants are unknown. Among them are the non-split **HSpin** groups. Most of the ingredients used in my split calculation already exist in the non-split case, except for Bermudez and Ruozzi's computation of the decomposable and indecomposable invariants of half-spin in [BR]. Bermudez and Ruozzi used the same techniques developed by Merkurjev in [Mer] where he computed invariants of non-split adjoint groups, and so it is expected that these techniques will also work in the non-split **HSpin** case. Once that computation is successfully completed, I expect that assembling the non-split ingredients will produce a description of the normalized invariants of $\mathbf{HSpin}(A, \sigma)$ for those algebras with orthogonal involution (A, σ) which admit a decomposition as a tensor product of smaller algebras. However, unlike for split (matrix) algebras, central simple algebras do not necessarily have a tensor decomposition this way. Therefore, a subsequent goal is then to understand the invariants of $\mathbf{HSpin}(A, \sigma)$ when there is not such a decomposition. This endeavour will likely require the development of novel techniques. Another problem which remains is to find an explicit description of the non-decomposable normalized invariants of **HSpin**. Indeed, this task is still outstanding for the split group $\mathbf{HSpin}_{4n}$ when $n \equiv 0 \pmod 4$. The pullback of an invariant of $\mathbf{PSO}_{4n}$ defined by Merkurjev describes those invariants which represent $2 \in \mathrm{Inv}^3(\mathbf{HSpin}_{4n})_{\text{norm}} \cong \mathbb{Z}/4\mathbb{Z}$, but the other invariants do not have a known explicit description. Finding these explicit descriptions is valuable for future study of higher degree invariants. The decomposable degree three invariants, those which can be constructed from degree two invariants, are critically important in understanding all degree three invariants. It is reasonable to expect that a similar subgroup of decomposable degree four invariants, which will be constructed from degree three invariants, will be just as useful. However, in order to make full use of these, the explicit behaviour of the lower degree invariants needs to be known.

References

[BR] H. Bermudez, A. Ruozzi, *Degree 3 Cohomological Invariants of Groups that are Neither Simply Connected nor Adjoint*, Journal of the Ramanujan Mathematical Society 29 (2013).

[CF] B. Calmès, J. Fasel, *Groupes classiques*, Autour des schémas en groupes, vol II, Panoramas et Synthèses **46** (2015), 1-133.

[EKLV] H. Esnault, B. Kahn, M. Levine, E. Viehweg, *The Arason Invariant and Mod 2 Algebraic Cycles*, Journal of the American Mathematical Society, Volume 11, Number 1 (1998), 73–118.

[Ford] T. J. Ford, *Separable Algebras*, Graduate Studies in Mathematics **183**, Amer. Math. Soc, Provident, RI, (2017).

[Gar] S. Garibaldi, *Orthogonal Involutions on Algebras of Degree 16 and the Killing Form of* E_8, Contemporary Mathematics 493, 2009.

[GMS] S. Garibaldi, A. Merkurjev, J.-P. Serre, *Cohomological Invariants in Galois Cohomology*, University Lecture Series 28, AMS, Providence, RI, 2003.

[GNR] P. Gille, E. Neher, C. Ruether, *Quadratic Pairs on Azumaya Algebras over a Scheme*. Preprint: https://arxiv.org/abs/2209.07107, September 2022, 45 pages.

[KMRT] M-A. Knus, A. Merkurjev, M. Rost, J-P. Tignol. *The Book of Involutions*. American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits.

[Mer] A. Merkurjev, *Degree Three Cohomological Invariants of Semisimple Groups*. J. European Math. Soc. 18 (2016), no. 2, 657–680.

[Rue] C. Ruether, *Rost Multipliers of Lifted Kronecker Tensor Products*, Documenta Mathematica 25:1997-2022, 2020.