Cohomological Obstructions to Quadratic Pairs over Schemes

Cameron Ruether
Joint with Philippe Gille and Erhard Neher

Memorial University of Newfoundland

AGAG, University of Virginia
May 27, 2023

$\mathrm{char}(\mathbb{F}) \neq 2$

$$\{\text{quadratic forms}\} \longleftrightarrow \{\text{Symmetric bilinear forms}\}$$

$$(q\colon V \to \mathbb{F}) \longmapsto b_q(x,y) = q(x+y) - q(x) - q(y)$$

$$q_b(x) = b(x,x) \longleftarrow\!\shortmid (b\colon V \times V \to \mathbb{F})$$

- $q = \frac{1}{2} q_{b_q}$ and $b = \frac{1}{2} b_{q_b}$.
- $\mathrm{PGO}^+(V,q) = \mathrm{PGO}^+(\mathrm{End}_{\mathbb{F}}(V), \sigma_q)$ when q is regular.
- $\mathrm{PGO}^+(A,\sigma)$ are smooth of type D.

Involutions

In any characteristic, let (A, σ) be a c.s.a. with $\mathbb{F}$-linear involution.

- σ is *orthogonal* if σ_{sep} on $A_{\mathsf{sep}} = \mathsf{M}_r(\mathbb{F}_{\mathrm{sep}})$ is adjoint to a symmetric bilinear form.
- σ is *weakly-symplectic* if σ_{sep} is adjoint to a skew-symmetric bilinear form.
- σ is *symplectic* if σ_{sep} is adjoint to an alternating bilinear form.

If $\mathrm{char}(\mathbb{F}) = 2$, $\mathrm{O}(A, \sigma)$ may not be smooth.

Quadratic Pairs

Definition

Let A be a central simple $\mathbb{F}$-algebra. A *quadratic pair* on A is (A, σ, f) where

- σ is an orthogonal involution on A, and
- $f : \mathrm{Sym}(A, \sigma) \to \mathbb{F}$ is a linear map satisfying

$$f(a + \sigma(a)) = \mathrm{Trd}_A(a)$$

for all $a \in A$. Here $\mathrm{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\}$.

- If $\mathrm{char}(\mathbb{F}) \neq 2$, $\Rightarrow f = \frac{1}{2}\,\mathrm{Trd}_{\mathcal{A}}$.
- The orthogonal group

$$\mathrm{O}(A, \sigma, f) = \{a \in A \mid \sigma(a) = a^{-1}, f(asa^{-1}) = f(s)\}$$

is smooth. $\mathrm{O}^+(A, \sigma, f)$ is semisimple type D.

Quadratic Pairs and Quadratic Forms

- If $A = \mathrm{End}_{\mathbb{F}}(V)$, then σ is adjoint to a regular $b\colon V \times V \to \mathbb{F}$.

$$(V \otimes_{\mathbb{F}} V, \mathrm{switch}) \xrightarrow{\sim} (\mathrm{End}_{\mathbb{F}}(V), \sigma)$$
$$x \otimes y \mapsto b(x, \_\_)y$$

Theorem (KMRT)

$\{$quadratic pairs involving $\sigma\} \leftrightarrow \{q$ whose polar is $b\}$

Idea: $f(x \otimes x) = q(x)$ and $f(x \otimes y + y \otimes x) = b(x, y)$

Classification

Let (A, σ) c.s.a. with orthogonal involution. Define

$$\begin{aligned} \mathrm{Sym}(A,\sigma) &= \{a \in A \mid \sigma(a) = a\} \\ \mathrm{Skew}(A,\sigma) &= \{a \in A \mid \sigma(a) = -a\} \\ \mathrm{Symd}(A,\sigma) &= \{a + \sigma(a) \mid a \in A\} \\ \mathrm{Alt}(A,\sigma) &= \{a - \sigma(a) \mid a \in A\} \end{aligned}$$

- The trace form

  $$\begin{aligned} A \times A &\mapsto \mathbb{F} \\ (a, b) &\mapsto \mathrm{Trd}_A(ab) \end{aligned}$$

  is a regular, symmetric, bilinear form.
- $\mathrm{Sym}(A,\sigma)^\perp = \mathrm{Alt}(A,\sigma)$ and $\mathrm{Alt}(A,\sigma)^\perp = \mathrm{Sym}(A,\sigma)$

Classification

Theorem (KMRT)

If (A, σ, f) is a quadratic pair on A, then there exists $\ell \in A$ such that

- $\ell + \sigma(\ell) = 1$,
- $f(s) = \mathrm{Trd}_A(\ell s)$,
- *ℓ is unique up to addition by an element from $\mathrm{Alt}(A, \sigma)$.*

Conversely, for any $\ell \in A$ satisfying $\ell + \sigma(\ell) = 1$, the linear map

$$\begin{aligned} f : \mathrm{Sym}(A, \sigma) &\to \mathbb{F} \\ s &\mapsto \mathrm{Trd}_A(\ell s) \end{aligned}$$

makes (A, σ, f) a quadratic pair. This form only depends on $[\ell] \in A/\mathrm{Alt}(A, \sigma)$.

Over a Scheme

Generalized by Calmès and Fasel.

- S is a fixed base scheme
- $\mathfrak{Sch}_S$ site with the fppf topology
- $\mathcal{O}$ the sheaf of rings $\mathcal{O}(T) = \Gamma(T, \mathcal{O}_T)$ for $T \in \mathfrak{Sch}_S$
- $\mathcal{A}$ an Azumaya $\mathcal{O}$–algebra of constant degree. So $\exists\{T_i \to S\}_{i \in I}$ such that

$$\mathcal{A}|_{T_i} \cong \mathrm{M}_r(\mathcal{O})|_{T_i}$$

Definition

Let $\mathcal{A}$ be an Azumaya $\mathcal{O}$–algebra. A *quadratic pair* on $\mathcal{A}$ is $(\mathcal{A}, \sigma, f)$ where

- σ is an orthogonal involution on $\mathcal{A}$, and
- $f : \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ is an $\mathcal{O}$–linear natural transformation satisfying

$$f(a + \sigma(a)) = \mathrm{Trd}_{\mathcal{A}}(a)$$

for all $T \in \mathfrak{Sch}_S$ and $a \in \mathcal{A}(T)$.

Question

- Given $(\mathcal{A}, \sigma)$, when can it be extended to $(\mathcal{A}, \sigma, f)$?
- Given $(\mathcal{A}, \sigma)$, what is a classification of all possible $(\mathcal{A}, \sigma, f)$?

$\mathrm{Id} + \sigma \colon \mathcal{A} \to \mathcal{A}$ and $\mathrm{Id} - \sigma \colon \mathcal{A} \to \mathcal{A}$. Define

- $\mathcal{S}ym_{\mathcal{A},\sigma} = \ker(\mathrm{Id} - \sigma)$
- $\mathcal{S}kew_{\mathcal{A},\sigma} = \ker(\mathrm{Id} + \sigma)$
- $\mathcal{S}ymd_{\mathcal{A},\sigma} = \mathrm{Im}(\mathrm{Id} + \sigma)$
- $\mathcal{A}\ell t_{\mathcal{A},\sigma} = \mathrm{Im}(\mathrm{Id} - \sigma)$

Over an Affine Scheme

If S is an affine scheme,

- $\mathcal{S}ym_{\mathcal{A},\sigma}, \mathcal{A}\ell t_{\mathcal{A},\sigma}$ are direct summands of $\mathcal{A}$ when σ is orthogonal, and mutually perpendicular w.r.t. the trace form.
- So, linear forms $f : \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ correspond to $\ell \in \mathcal{A}(S)$ with $\ell + \sigma(\ell) = 1$ up to addition by an element of $\mathcal{A}\ell t(\mathcal{A}, \sigma)$.

If S is any scheme, and $(\mathcal{A}, \sigma, f)$ a quadratic pair

- $\{U_i \to S\}_{i \in I}$ an affine open cover
- $(\mathcal{A}|_{U_i}, \sigma|_{U_i}, f|_{U_i})$ will be given by some $\ell_i \in \mathcal{A}(U_i)$ with $\ell_i + \sigma(\ell_i) = 1$.
- $\Rightarrow 1 \in \mathcal{S}ymd_{\mathcal{A},\sigma}(U_i)$ for all $i \in I$,
- $\Rightarrow 1 \in \mathcal{S}ymd_{\mathcal{A},\sigma}(S)$.

Locally Quadratic Involutions

Definition

We call $(\mathcal{A},\sigma)$ an Azumaya $\mathcal{O}$–algebra with *locally quadratic involution* if $1 \in \mathit{Symd}_{\mathcal{A},\sigma}(S)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathit{Skew}_{\mathcal{A},\sigma} & \hookrightarrow & \mathcal{A} & \xrightarrow{\mathrm{Id}+\sigma} & \mathit{Symd}_{\mathcal{A},\sigma} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathit{Skew}_{\mathcal{A},\sigma}/\mathcal{A}\ell t_{\mathcal{A},\sigma} & \hookrightarrow & \mathcal{A}/\mathcal{A}\ell t_{\mathcal{A},\sigma} & \xrightarrow{\pi} & \mathit{Symd}_{\mathcal{A},\sigma} & \longrightarrow & 0
\end{array}$$

Cohomological Obstructions

$$
\begin{array}{ccccccc}
\mathcal{A}(S) & \longrightarrow & \mathcal{S}ymd_{\mathcal{A},\sigma}(S) & \longrightarrow & H^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}) & \longrightarrow & \dots \\
\downarrow & & \| & & \downarrow & & \\
(\mathcal{A}/\mathcal{A}\ell t_{\mathcal{A},\sigma})(S) & \xrightarrow{\pi(S)} & \mathcal{S}ymd_{\mathcal{A},\sigma}(S) & \longrightarrow & H^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}\ell t_{\mathcal{A},\sigma}) & \longrightarrow & \dots
\end{array}
$$

$$
\begin{array}{ccc}
1 & \longmapsto & \Omega(\mathcal{A},\sigma) \\
\| & & \downarrow \\
1 & \longmapsto & \omega(\mathcal{A},\sigma)
\end{array}
$$

We call $\Omega(\mathcal{A},\sigma)$ the *strong obstruction* and $\omega(\mathcal{A},\sigma)$ the *weak obstruction*.

Cohomological Obstructions

Theorem (Gille, Neher, R.)

- $\Omega(\mathcal{A},\sigma) = 0 \Leftrightarrow \exists f = \mathrm{Trd}_{\mathcal{A}}(\ell\_\_)$ *with* $\ell + \sigma(\ell) = 1$ *for* $\ell \in \mathcal{A}(S)$.
- $\omega(\mathcal{A},\sigma) = 0 \Leftrightarrow \exists f : \mathcal{S}ym_{\mathcal{A},\sigma} \to \mathcal{O}$ *making* $(\mathcal{A},\sigma,f)$ *a quadratic pair.*
- *There is a classification*

$$\{f \text{ extending } (\mathcal{A},\sigma)\} \leftrightarrow \pi(S)^{-1}(1) \subset (\mathcal{A}/\mathcal{A}\ell t_{\mathcal{A},\sigma})(S).$$

Proof.

- Any $(\mathcal{A},\sigma,f)$ is given locally by ℓ_i, which glue to a section $\lambda \in (\mathcal{A}/\mathcal{A}\ell t_{\mathcal{A},\sigma})(S)$.
- Any global section λ is locally given by ℓ_i from $\mathcal{A}$, the $f_i = \mathrm{Trd}_{\mathcal{A}}(\ell_i\_\_)$ glue into a global f.
- $\omega(\mathcal{A},\sigma) = [\pi^{-1}(1)] \in H^1(S, \mathcal{S}kew_{\mathcal{A},\sigma}/\mathcal{A}\ell t_{\mathcal{A},\sigma})$ where

$$\pi^{-1}(1) \colon \mathfrak{Sch}_S \to \mathfrak{Sets}, \quad \pi(T)^{-1}(1|_T)$$

An Example

Let $\mathrm{char}(\mathbb{F}) = 2$. Take E an ordinary elliptic curve as the base scheme.

$$E \xrightarrow{\cdot 2} E$$

is an $E[2] = \mu_2 \times_{\mathbb{F}} \mathbb{Z}/2\mathbb{Z}$ torsor.

$$\mu_2 \times_{\mathbb{F}} \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathrm{PGL}_2$$

$\Rightarrow E \wedge^{\mu_2 \times_{\mathbb{F}} \mathbb{Z}/2\mathbb{Z}} \mathrm{PGL}_2$ is a PGL_2–torsor, so defines $\mathcal{Q}$ a quaternion algebra. It has canonical involution $(\mathcal{Q}, \sigma)$.

- σ is symplectic, hence also orthogonal.
- σ can be extended to a quadratic pair ($\Rightarrow$ locally quadratic). $\Rightarrow \omega(\mathcal{Q}, \sigma) \neq 0$.
- $\mathcal{Q}(E) = \mathbb{F}$. So $\ell + \sigma(\ell) = 2\ell = 0$ for all $\ell \in \mathcal{Q}(E)$. $\Rightarrow \Omega(\mathcal{Q}, \sigma) = 0$.

Another Example

$\mathrm{char}(\mathbb{F}) = 2$, $\mathbb{F} = \overline{\mathbb{F}}$. Let $\Gamma = \mathrm{PGL}(\mathbb{F}_4)$ as an abstract group, $\Gamma_{\mathrm{Spec}(\mathbb{F})}$ the constant group scheme.
Serre: $\exists\ Y \to S$ a Γ–cover between smooth projective $\mathbb{F}$-varieties. This is a $\Gamma_{\mathrm{Spec}(\mathbb{F})}$–torsor.

$$\Gamma_{\mathrm{Spec}(\mathbb{F})} \hookrightarrow \mathrm{PGL}_2$$

$\Rightarrow Y \wedge^{\Gamma_{\mathrm{Spec}(\mathbb{F})}} \mathrm{PGL}_2$ is a PGL_2–torsor, so defines $(\mathcal{Q}, \sigma)$ a quaternion algebra with symplectic/orthogonal involution.

- σ is locally quadratic.
- $(\mathcal{Q}, \sigma)$ splits over Y.
- If we had $(\mathcal{Q}, \sigma, f)$, then $f|_Y : \mathcal{S}ym_{\mathrm{M}_2(\mathcal{O})|_Y, \sigma|_Y} \to \mathcal{O}|_Y$ must be Γ–equivariant.
- $\Rightarrow f = 0$, contradiction.
- So $\omega(\mathcal{Q}, \sigma) \neq 0$. ($\Rightarrow \Omega(\mathcal{Q}, \sigma) \neq 0$.)

Thank You